

A new system of coordinates for the tilings $\{p, 3\}$ and $\{p-2, 4\}$.

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Abstract

In this paper, a new way to define coordinates for the tiles of the tilings $\{p, 3\}$ and $\{p-2, 4\}$ where the natural number p satisfies $p \geq 7$ is investigated.

1 Introduction

In [4], a system of coordinates for the tilings $\{p, 3\}$ and $\{p-2, 4\}$ was proposed. In this paper, we revisit this question. Several points were not enough investigated in [4]. As I needed an exact account for a result which will appear in a forthcoming paper, I thought it useful to provide this exact description which is of interest in itself. It is based in a new look with respect to what was written in [4].

We refer the reader to [4, 6] for an introduction to an algorithmic approach to the tessellations of the hyperbolic plane. The first section, introduces the leftmost approach and the one based on the preferred son as defined in [4]. The novelty consists in comparing these approaches which allows us to define the properties of both constructions in a simple way. In a preliminary Section 2, direct proofs are given for establishing the bijection of a sector of the tessellations $\{p-2, 4\}$ and $\{p, 3\}$ with a tree. Section 3, page 10, defines both just mentioned approaches. The consequences are gathered and proved in Section 4, page 24, giving the coordinates of the neighbours of a node in terms of the coordinate of the node itself.

2 The bijection with a tree

We remind the reader that a **sector** of the tessellations we consider is the set of tiles whose centre is inside the angle defined by two rays meeting at a point. In the case of the tilings $\{p-2, 4\}$, where $p \geq 7$, we assume that the rays meet at a vertex V of a tile T and that they support consecutive sides of T . The sector contains T , V is called its **vertex** and the rays its **borders**. Assume that u is

before v in the sector while counter-clockwise turning around V . We say that u , v is the **first**, *second* border respectively. In the case of $\{p, 3\}$, we assume that the rays meet at a midpoint M of a side σ belonging two tow tiles T_1 and T_2 . Fix an end V of σ . Let T_3 be the third tile sharing V with T_1 and T_2 . We assume that one of the rays, say u , also passes through the mid-point of the side shared by T_1 and T_3 and that the other, say v , also passes through the mid-point of the side shared by T_2 and T_3 . We assume that the centre of T_3 is contained in the angle defined by u and v at M . The sector is defined in the same way as previously: it is the set of tiles whose centre lies in the angle defined by u and v which are again called the borders of the sector.

In [7], a proof that a sector of the pentagrid is in bijection with a tree can be found. The left-hand side part of Figure 1 illustrates its idea. The proof can be adapted to the situation of the heptagrid as shown by Figure 3. In Subsections 2.1 and 2.2, we extend these results to the tessellations $\{p-2, 4\}$ and $\{p, 3\}$ respectively, where $p \geq 7$ in both cases. Figure 1 illustrates the proof also in the case of the tessellation $\{6, 4\}$ while Figure 3 does the same for the tessellation $\{9, 3\}$.

In the next lemmas, we shall use a tool which we call the **numbering** of the sides of a tile T . We fix a side σ of T . Starting from σ and counter-clockwise turning around T , we increasingly assign a number to the sides, assigning 1 to σ . We call the process a numbering which is defined once its side 1 is fixed. We also number the line which supports the side i by i . We also number the vertices as follows: the vertex $i+1$ is shared by the sides i and $i+1$ for $i \in \{1..p-1\}$ and vertex 1 is shared by the sides p and 1.

We remind here the elementary properties of a regular convex polygon P of the hyperbolic plane. The interior angle 2α is fixed. There is a single isosceles triangle T_1 whose basis angle is α and its vertex angle is $\frac{2\pi}{p}$. Let O be the vertex of T . Replicating $p-1$ rotations around O produces a copy Q of the polygon. Accordingly, the circle around O which passes through a vertex of Q also passes through the other vertices of Q , it is the **circumscribed** circle of Q . From this, we get that the diameter which passes through a vertex or through a mid-point, we get the same diameters when p is odd, defines a reflection which leaves Q globally invariant.

2.1 Tessellations $\{p-2, 4\}$

Consider a tile T . The complement in the plane of the lines supporting its sides defines $2p-4+1$ regions which are pairwise disjoint. One region is bounded: it is inside T , the others are infinite. Fix a numbering of T . Call region i_1 the region which is in contact with the side i . Call region i_2 the region which is in contact with the vertex shared by the side i and the side $i+1$ for $i < p$ and the side p and side 1. The regions i_1 , i_2 are called of type 1, type 2 respectively.

Consider a region of type 1. We may assume that it is region 1_1 . The region is delimited by the lines 1, 2 and p . A line i defines two half-planes. Call **inside** of i , **outside** of i the half-plane defined by the line i which contains T , does

not contain T respectively. The region 1_1 is inside the lines 2 and p and it is outside the line 1. It is plain that the other regions of type 1 are constructed in the same way. Consider the region 1_2 of type 2: it is the intersection of the outside of line 1 and the outside of line 2.

LEMMA 1 *Consider a sector S of the tessellation $\{p-2, 4\}$. Let T be a tile whose centre is inside S . Fix a numbering of the sides of T . The lines i with $3 \leq i \leq p-3$ are non secant with line 1. Say that a point P is **visible** from a side σ of T if P and the centre of T are on the same side of σ . If P is in a region of type 1, type 2, it is not visible from the side, the two sides respectively which are in contact with this region and it is visible from all the other sides.*

Proof. Consider the numbering indicated by the lemma. As side 2 is orthogonal to both side 1 and side 3, the lines 1 and 3 are non-secant. Consider the rays b_i issued from the centre of T , meeting the side i and supported by the bisector of the side i . Let B_i be the sector whose vertex is the centre of T which is delimited by the rays b_{i+1} and b_{i-1} in this order, with $2 \leq i \leq p-2$ and B_1 being delimited by b_{p-2} and b_2 in this order. The angle between the rays is $\frac{4\pi}{p}$. As the side i is both orthogonal to b_i and to the lines $i-1$ and $i+1$, where $i-1$ is replaced by $p-2$ when $i = 1$, these two lines are non-secant with b_i . Consequently, the line i is completely contained in B_i . Now, by an angle argument, it is plain that B_i is disjoint from B_1 if and only if $3 \leq i \leq p-3$. This proves that the line i is non secant with line 1 for those values of i .

Consider a point P which is outside T . It is in a single region i_1 or i_2 . We may assume that it is the region 1_1 or 1_2 by changing the side 1 of the numbering.

First, assume that P is in the region 1_1 . For the values of i such that $3 \leq i \leq p-2$, as the considered sectors defined by b_i and b_{i+2} are disjoint from that defined by b_p and b_2 , the inside of the line i also contains P . For the sides 2 and $p-2$, by definition of the types of the regions, P is also in the inside of the lines both for line 1 and for the line $p-2$.

Secondly, assume that P is in the region 1_2 . Then P is in the outside of line 2 and still in the inside of the line $p-2$ and in the inside of the lines i for $3 \leq i \leq p-3$. This proves the lemma. \square

From the lemma, we can prove another property :

LEMMA 2 *Let S be a sector of the tessellation $\{p-2, 4\}$. Let T be a tile whose centre is in S . Let P be a point outside T such that its orthogonal projection on the line 1 of T falls inside the side 1 of T . The orthogonal projection of P on the lines i with $i \in \{3..p-3\}$ also falls inside the side i .*

Proof of Lemma 2. Indeed, let H be the orthogonal projection of P on the line 1 of T . From the hypothesis, H is inside the side 1 of T . Let K be the projection of P on the line i with $i \in \{3..p-3\}$. The line PK does not meet the line $i \ominus 1$: otherwise, let L be the intersection with the line $i \ominus 1$. As the sides i and $i \ominus 1$ are perpendicular, from the intersection, there would be two distinct

perpendiculars to the line i unless P lies on the side $i \ominus 1$ and H belongs to the line PK . But we assumed that H is inside the side 1 of T . A similar argument with the line $i \oplus 1$ which is also orthogonal to the line i shows us that K is inside the side i . This proves the lemma. \square

Note that if P has its orthogonal projection on the line 1 falls inside the side 1 of T , its orthogonal projection on the line 2 cannot fall inside the side 2 of T as the line 2 is orthogonal to the line 1 of T . The same remark also holds for the line $p-2$.

We are now in the position to prove the following result:

THEOREM 1 *The set of tiles of a sector S of the tessellation $\{p-2, 4\}$ is in bijection with a tree possessing to kinds of nodes, B and W , generated by the following two rules: $W \rightarrow BW^{p-5}$ and $B \rightarrow BW^{p-6}$, the root being a W -node.*

Proof of Theorem 1. Let \mathcal{H} be the tile of S which has the vertex of S among its vertices. Call it the **head** of S . Let O be the centre of \mathcal{H} , and let be u and v the rays defining S , the first and the second borders of \mathcal{H} respectively. We define the **cornucopia** of the sector as the set of tiles in the sector which share a side with the first border.

We attach the root of the tree to \mathcal{H} . Number the sides of \mathcal{H} with its side on u as its side 1. Let T_j , with $j \in \{2..p-3\}$, be the reflection of \mathcal{H} in its side j . Let H_j be the orthogonal projection of O on the line j of \mathcal{H} . As \mathcal{H} is a regular convex polygon, H_j is the midpoint of the side j of \mathcal{H} . Define S_3 as the image of S by the shift along the side 2 of \mathcal{H} . For $k \in \{3..p-4\}$, define S_{k+1} as the image of S_k under the rotation around O which transforms the side k into the side $k+1$. Now, the complement of \mathcal{H} in S can be decomposed into the $p-5$ sectors S_k with $k \in \{3..p-3\}$ together with a remaining region which we call a **strip**, denote it by B . The head of S_k is T_k as can easily be seen. Each T_k is numbered with, as its side 1, the side k of \mathcal{H} . As each S_k with $k \in \{3..p-3\}$ is in the outside of the line k of \mathcal{H} , the distance from O to S_k is at least that from O to the line k , so that it is OH_k , as H_k is the orthogonal projection of O on the line k . Denote by C_k , with $k \geq 1$, the tiles of the cornucopia, with $C_1 = \mathcal{H}$, C_2 being its image by reflection in the side 2 of C_1 . Fix the side 1 of C_2 to be the side 2 of \mathcal{H} . Then, for $k \in \{2..p-4\}$ C_{k+1} is the image of C_k by reflection on its side 3, the side 1 of C_{k+1} being the side 3 of C_k . We note that C_2 is in B : we call it the head of B which is delimited by the side 1 of C_2 , the ray u supporting the side 1 of C_1 , and the first border of S_3 , which supports the side $p-2$ of C_2 .

Denote by S_u the image of S by the shift along u which also transforms C_1 into C_2 . Now, if we repeat to S_u the process which was performed in S to define the S_k 's starting from S , we obtain a sequence of sectors Q_j with $j \in \{3..p-4\}$ where Q_3 is the image of S_u by the shift along the side 3 of C_2 and then Q_{j+1} is the the image of Q_j by the rotation around the centre of C_2 which transforms its side j into its side $j+1$. For the numbering of Q_j , we fix its side 1 as the side j of C_2 . The complement of C_2 in the strip B consists of the Q_j 's for $j \in \{3..p-4\}$ and a new strip which is the image of B under the shift which transforms S into S_u . As $B \subset S_u$, the distance from O to B is at least that

from O to S_u which, clearly, is OH_2 which is also the distance from O to C_2 . We say that C_2 is the B son of \mathcal{H} and that the T_k 's with $k \in \{3..p-3\}$ are its W -sons. The sons of \mathcal{H} constitute the level 1 of the tree. The sons of C_2 are C_3 , the B -son, and the heads of the Q_j 's with $j \in \{3..p-4\}$, the W -sons.

From now on, say that the head of a strip is a B -node, and that the head of a sector is a W -node. Define their **sons** as their image by reflection in the sides j with $j \in \{2..p-3\}$ for W -nodes and in both $j \in \{3..p-3\}$ for B -nodes. The reflection in the side 2, side 3 yields the B -node for W -, B -nodes, respectively. Recursively repeating this decomposition can be represented by a recursive application of the rules given in the theorem which provides us with the tree stated in the theorem. This defines an injection of the tree into S .

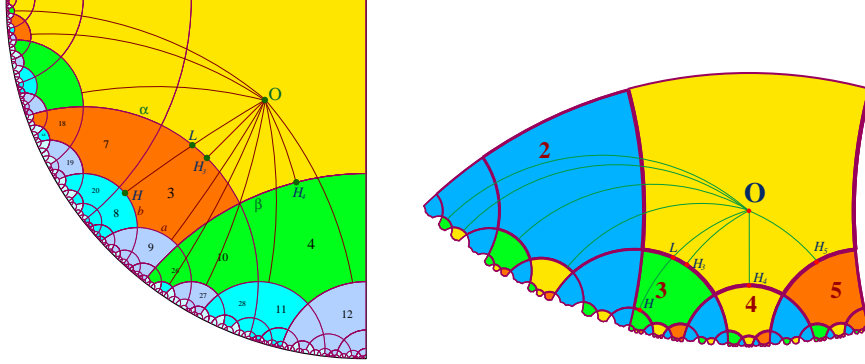


Figure 1 *Proof of the bijection for the pentagrid, to left, and for the tessellation $\{6,4\}$, to right. It can be generalized to the tessellations $\{p-2,4\}$.*

We proved that the distance from O to B is OH_2 and that the distance from O to S_j for $j \in \{3..p-3\}$ is OH_j . As O is the centre of \mathcal{H} , all these distances are equal, denote by a their common value. From the previous construction, it is plain that the tree is injectively mapped into the sector by the correspondence which associates a tile to a node.

We say that B and the S_j 's we constructed from \mathcal{H} define the first generation. Applying the just above defined construction to the generation n , the sons of the nodes of the generation n constitute the generation $n+1$. Note that, in our construction, O is visible from the side 1 of the white sons of \mathcal{H} . Consider a white node T of the generation n and assume that O belongs to the region R of type 1 associated to the side 1 of T , say σ_0 . From Lemma 1, we obtain that O is visible from the sides 1 of the W -sons of T . Accordingly, Lemma 2 says that the orthogonal projection of O on the side 1 of a W -son of T lies inside that side. Fix a W -son of T and let σ_1 be its side 1. Denote by H_0 , H_1 the orthogonal projection of O on σ_0 , σ_1 respectively. Lemma 2 says that H_0 , H_1 is inside σ_0 , σ_1 respectively, see Figure 1. As H_1 and O are not on the same side of the line ℓ which supports σ_0 , OH_1 cuts ℓ at L . Clearly, $OL \geq OH_0$ as

OH_0 is the distance from O to ℓ_0 . Let ℓ_1 be the line supporting σ_1 . From Lemma 1, ℓ_1 and ℓ_0 are non-secant. Accordingly, there is a point P_0 on ℓ_0 and a point P_1 on ℓ_1 such that $\ell_0 \perp P_0P_1$ and $\ell_1 \perp P_0P_1$. Hence, P_0P_1 realizes the distance between ℓ_0 and ℓ_1 , so that $H_0L \geq P_0P_1$. If b is the smallest value between a and the distances d_j between the line 1 of T and the lines j of T with $j \in \{3..p-3\}$ which are non secant with the line 1 from Lemma 1, we get that $OH_1 \geq OH_0 + b$. Accordingly, if we assumed that $OH_0 \geq nb$, we get that $OH_1 \geq (n+1)b$. Now, the same arguments can be repeated with the sons of C_2 as O is visible from its sides j with $j \in \{3..p-3\}$.

To get the bijection property, it is enough to prove that any point in the angle defined by the vertex of S and its borders and by the fact it contains \mathcal{H} , belongs to at least one tile T of S . Let P be such a point. If P belongs to the cornucopia, as the distance from C_k to O is at least $(k-1)a$, it falls in at least on of the C_k 's. Note that the C_k 's are successive B -sons of B -sons of C_1 . If not, from the above construction, P belongs to one of the S_j 's with $j \in \{3..p-3\}$ headed by the W -sons of the level 1 of the tree, or to the W -sons of C_k with $k \geq 2$ which belong to the level k . Note that from what we proved, the strip whose head is on the level n is at a distance at least nb from O . Let S^1 be the sector in which P lies. We repeat the same argument. If P is in the cornucopia K_1 of S^1 , we shall find it in a tile of K_1 , otherwise it will be in a sector S^2 obtained from the process we just described. By our arguments, the distance from O to S^1 , S^2 is at least n_1b , n_2b for some positive integer n_1 , n_2 respectively and, our argument proves that $n_2 > n_1$. We can construct a sequence S^1, S^2, \dots, S^m containing P . From this observation, as OP is finite, there is an m such that P is contained in the cornucopia of S^m . Consequently, we shall find a tile of S containing P and this tile, by our above observation, will be in correspondence with a node of the tree. This completes our proof. \square

From the bijection property, we shall say indifferently tile or node for a tile. When we shall use the term node, we shall make it precise whether it is a B - or a W -node if it is needed. From the proof of the theorem, we conclude the following property:

COROLLARY 1 *The B -nodes of the tree are in the cornucopias. The head of the cornucopias are the W -nodes exactly.*

2.2 Tessellations $\{p, 3\}$

Consider a tile T of the tessellation $\{p, 3\}$ with $p \geq 7$. We number the vertices of T such that the vertices i and $i+1$ are the ends of the side i when $i < p$ and the vertices p and 1 are the ends of the side p , see Figure 2 where the numbers are written only for vertex 1.

Consider the mid-point line 1 which, by definition, joins the midpoints of the sides p and 1. This name is grounded on the following considerations. Let M_i be the mid-point of the side i and denote by V_i the vertex i . Let T_1 be the reflection of T in its side 1. Let N_2 be the mid-point of the side of T_2 which abuts the vertex 2 of T . Then the triangles $N_2V_2M_1$ and $M_1V_1M_p$ are isosceles

triangles as the interior angles at V_2 in T_2 and V_1 in T are equal. Then, the basis angles of these triangles are equal and as N_2 and M_2 are not on the same side of line 1 as far as T_2 is the reflection of T in this line, this means that the points N_2 , M_1 and M_p lie on the same line which we call mid-point line 1. Also note that the considered triangles being isosceles, the basis angle is acute.

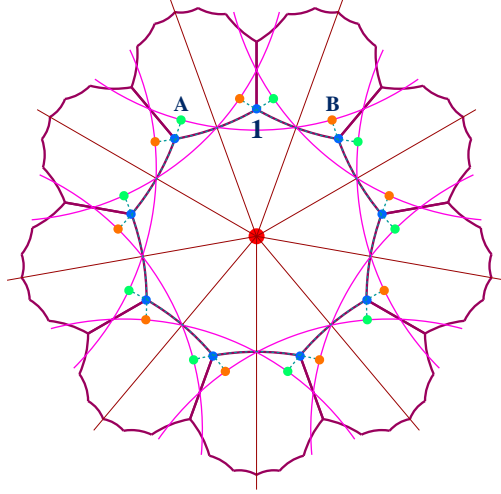


Figure 2 The pseudo-sides in a tile of the tessellation $\{p, 3\}$.

Let O be the centre of T . Note that OM_1 is a symmetry axis of T which leaves the side 1 globally invariant, so that $OM_1 \perp V_1V_2$. As the basis angle (M_1V_1, M_1M_p) is acute, V_1 and O are not on the same side of M_1M_p . Note that OV_1 is also a symmetry axis of T which leaves vertex 1 invariant. The symmetry exchanges V_2 and V_p so that it also exchanges M_1 and M_p . Consequently, $M_1M_p \perp OV_1$. We say that the **inside** of the mid-point line 1 is its half-plane which contains the centre of T . At last, note that line 2 cuts mid-point line 1 at a point A and the line $p-2$ cuts mid-point line 1 at B , in both cases at a right-angle. We call AB the **pseudo-side** 1 of T which, consequently, is supported by mid-point line 1. From our last observation, the mid-point lines i and $i+2$ with $1 \leq i < p-1$ are non secant. The mid-point lines 1 and $p-2$ are also non-secant. Also note that the reflection in OM_2 exchanges the mid-point lines 1 and 3. As side 2 is also perpendicular to OM_2 , that line is non-secant with both mid-point lines 1 and 3. From these considerations, we also get that the mid-point line 1 is completely included in the the angle (OM_2, OM_{p-1}) .

By the symmetries of T which is invariant under any rotation around O which transforms a side of T into a side of T , these just mentioned properties can be transported to any vertex of T and to any mid-point line which are numbered as just mentioned. In order to facilitate the notations, we introduce

two operations:

$$\begin{aligned} i \oplus j &\text{ is } i+j \text{ if } i+j \leq p, \text{ otherwise it is } i+j-p; \\ i \ominus j &\text{ is } -i-j \text{ if } i-j \geq 1, \text{ otherwise it is } i-j+p \end{aligned}$$

We are now in the position to prove for the tessellation two lemmas which are analogous to Lemmas 1 and 2. We have a different notion of region as in the case of the tessellations $\{p-2, 4\}$. Around T , we consider the regions R_i which are the intersections of the outside of the pseudo-side i with the inside of the lines $i \ominus 2$ and $i \oplus 1$. Define μ_i to be the bisector of the side i . Then the mid-point line i and $\mu_{i \oplus 1}$ are both perpendicular to line $i \oplus 1$ so that, $\mu_{i \oplus 1}$ is non-secant with the mid-point line i . Similarly, $\mu_{i \ominus 2}$ is non-secant with the mid-point line i as both those lines are perpendicular to line i . Accordingly, the outside of the mid-point line i is contained in the angle at O defined by $(\mu_{i \ominus 2}, \mu_{i \oplus 1})$ whose measure is $\frac{6\pi}{p}$. For $p \geq 7$, this angle is less than π .

Let U be a vertex of a tile of the tessellation $\{p, 3\}$. Three sides s_1, s_2 and s_3 meet at U , pairwise belonging to the three tiles T_1, T_2 and T_3 which meet at V . Assume that s_i is shared by T_j and T_k where $\{i, j, k\} = \{1, 2, 3\}$. We may assume that the order s_1, s_2, s_3 define the counter-clockwise orientation. It is then the same for the tiles T_i . Let V be the mid-point of s_1 . Let u, v be the ray issued from V which is supported by the mid-point line passing through V and through the mid-point of s_2, s_3 respectively. Then we define the **sector** S of the tessellation $\{p, 3\}$ defined by V, u and v as the set of tiles whose centers are inside the angle (u, v) . Note that T_1 belongs to S . We say that u is the **first** border of S , that v is its **second** one.

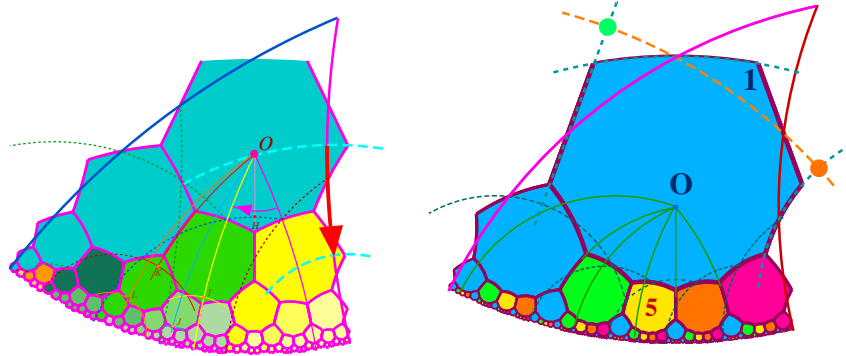


Figure 3 Proof of the bijection for the tilings $\{p, 3\}$. To left, the heptagrid, to right, an illustration for the general case with the tessellation $\{9, 3\}$.

LEMMA 3 Let T be a tile of the tessellation $\{p, 3\}$. The mid-point lines i of T with $i \in \{4..p-2\}$ are non secant with the mid-point line 1.

Proof of Lemma 3. This comes from the above remark about the fact that the mid-point line i is completely included in the angle defined by $(\mu_{i \ominus 2}, \mu_{i \oplus 1})$, with

the additional property that that mid-point line is non secant with both $\mu_{i\oplus 2}$ and $\mu_{i\oplus 1}$. \square

We are now in the position to prove the following result,

THEOREM 2 *The set of tiles of a sector S of the tessellation $\{p, 3\}$ is in bijection with a tree possessing to kinds of nodes, B and W , generated by the following two rules: $W \rightarrow BW^{p-5}$ and $B \rightarrow BW^{p-6}$, the root being a W -node. It is the same tree as the tree defined in Theorem 1 for the tessellation $\{p-2, 4\}$.*

Proof of Theorem 2. We follow the same line as in the case of Theorem 1. Figure 3 illustrates the principle we apply, to the left-hand side of the figure, in the heptagrid, to its right-hand side, to the tessellation $\{9, 3\}$. Let u, v be the first, second respectively border of S and let V be its vertex. Consider \mathcal{H} the head of S . Let vertex 1 of T be its closest vertex to V . Denote by T_i with $i \in \{3..p-2\}$ the reflection of \mathcal{H} in its side i . Fix the side 1 of T_i as the side i of \mathcal{H} . Then the vertex 1 of T_i is the vertex $i \oplus 1$ of \mathcal{H} . Then, the shift along v which transforms the vertex 1 of \mathcal{H} into its vertex $p-1$ also transforms \mathcal{H} into T_{p-2} . It also transforms S into a sector S_{p-2} whose vertex is the mid-point of the side $p-1$, whose first border is supported by the mid-point line $p-1$ of \mathcal{H} and its second one is v . Now, the successive rotations around O , the centre of \mathcal{H} , transform S_{p-2} into S_{p-j} with $j \in \{3..p-3\}$. Considering the complement in S of \mathcal{H} and of the S_i with $i \in \{4..p-2\}$, we get a region which we again denote by B and which we call again a **strip**. Note that $B \subset S_3$, the inclusion being proper.

The strip B , is delimited by u , by the pseudo-side 4 and the mid-point line 5. We call T_2 the **head** of B . The shift along the mid-point line 5 of \mathcal{H} transforms the vertex 1 of T_2 into its vertex $p-1$, hence the sector S_3 into a sector Q_{p-2} . Similarly, successive rotations around the centre of T_2 transform Q_{p-2} into Q_{p-j} with $j \in \{3..p-4\}$. The head Y_i of Q_i is obtained by the reflection of T_2 in its side i when $i \in \{4..p-2\}$. It can be noted that the complement of T_2 in B and of the Q_i 's with $i \in \{5..p-2\}$ is again a strip, the image of B in the shift along u which transforms \mathcal{H} into T_2 . Here too, we fix that the side 1 of Y_i for $i \in \{4..p-2\}$ as the side i of T_2 . Now, we can repeat the argument of Theorem 1: recursively repeating this construction with sectors and strips, we obtain an injection from the tree defined in the statement of Theorem 1 into the tiles of S . Again, we have to prove that the mapping is surjective.

We proceed as in the proof of Theorem 1, using the mid-point lines in order to estimate the distance from O to a region defined by the above process. We note that, from the above discussion leading to Lemma 3, for each i , the vertex i of a tile T is not in the same side as the centre of T with respect to the mid-point line i . The proof of the mid-point line property show us that for any i , the centres of T_i and $T_{i\oplus 1}$ are outside the mid-point line i of \mathcal{H} . The distance from O to these centres can be estimated by the distance from O to the mid-point line i of T_i . Accordingly, this distance estimates the distance from O to both S_i and $S_{i\oplus 1}$. Note that the mid-point line 3 of \mathcal{H} allows us to estimate the distance from O to B .

As previously, we define the **cornucopia** of S as the set of its tiles which share a side with u and \mathcal{H} is called the head of the cornucopia. The complement of the cornucopia in S can be split as a union of sectors which, sectors are set of tiles, are pairwise disjoint. This construction can be repeated for all sectors.

Presently, assume that the distance from O to any sector or strip whose head belongs to the level n of the tree is $n.a$, where a is the smallest distance from mid-point line 1 to the mid-point lines i of the same tile which are non secant with it. Let T be the head of a sector or a cornucopia which is a node of the level $n+1$. Let H be the orthogonal projection of O in the mid-point line 1 or 2 of T , depending on the position of T with respect to its father U in the tree, call ℓ this line. If U is the head of a sector S_0 or of a cornucopia, its distance from O is measured by the orthogonal projection of O on the mid-point line 1 or 2 of U , denote it by μ . Now, from the construction, the centre of T is not on the same side of μ as O . Accordingly, OH cuts μ at L . Now, by induction, as U belongs to the level n , $OL \geq na$. Also, as ℓ and μ are non-secant, $LH \geq a$. Accordingly $OH \geq (n+1)a$, which completes the proof of the theorem. \square

3 The leftmost and the preferred son approaches

From Theorems 1 and 2, we know that a sector of the tessellation $\{p-2, 4\}$ or one of the tessellation $\{p, 3\}$ with the same value of p , $p \geq 7$, are both in bijection with the same tree. Now, taking into consideration that the tree is embedded in the dual graph of the tilings, we shall see that we can define infinitely many trees which are in bijection with any sector of those tessellations. Subsections 3.3, and 3.2 define these trees. Subsection 3.3 is based on the construction performed in Subsection 3.2. But the definition of one the trees relies on a notion we take from the property stated in Theorems 1 and 2. We turn to this point in Subsection 3.1.

3.1 The coordinates of the nodes

In both Theorems 1 and 2, the tree has two kinds of nodes, B -nodes and W -nodes, and it is constructed by the recursive application of the following rules:

$$\begin{aligned} B &\rightarrow BW^{p-6} \\ W &\rightarrow BW^{p-5} \end{aligned} \tag{1}$$

Note that we can also associate to (1) the matrix $\begin{smallmatrix} p-6 & 1 \\ p-5 & 1 \end{smallmatrix}$. From this, if w_n , v_n is the number of W -nodes, B -nodes respectively on the level n of the tree, we get that:

$$\begin{aligned} v_0 &= 0, w_0 = 1 \\ v_{n+1} &= v_n + w_n \\ w_{n+1} &= (p-6)v_n + (p-5)w_n \end{aligned} \tag{2}$$

Denote by u_n the number of nodes on the level n . As each node produces one B -son exactly, $v_{n+1} = u_n$, so that, summing the last two lines of (2), we obtain:

$$\begin{aligned} u_{-1} &= 0, u_0 = 1 \\ u_{n+2} &= (p-4)u_{n+1} - u_n \end{aligned} \quad (3)$$

From that, we derive a matrix: $\begin{pmatrix} p-4 & -1 \\ 1 & 0 \end{pmatrix}$ whose characteristic polynomial is $P(X) = X^2 - (p-4)X + 1$. This polynomial is also the characteristic polynomial of the matrix associated to the rules (1). The polynomial has two positive roots, the greatest one is $\beta = \frac{(p-4) + \sqrt{(p-4)^2 - 4}}{2}$. As $p \geq 7$, $(p-4)^2 > 4$, so that β is a real number and $\beta > 1$ as $\frac{p-4}{2} > 1$. We easily check that $p-5 < \beta < p-4$ as $P(p-5) = 6-p \leq -1$ and as $P(p-4) = 1$. We set $b = p-4$ and $\mathbf{b}_1 = b-1$. It is known, see [1, 2, 3], that we can represent any natural number n as a sum of terms of the sequence defined by equation (3) :

$$n = \sum_{i=1}^k \alpha_i, \text{ where } \alpha_i \in \{0..b_1\}. \quad (4)$$

This sum can be formally represented in a position numeral system by $\alpha_k.. \alpha_1 \alpha_0$. We shall write: $[n] = \alpha_k.. \alpha_1 \alpha_0$, where the α_i 's are defined by (4) and we shall write $[\alpha_k.. \alpha_1 \alpha_0] = [[n]] = n$ for the converse operation. Note that the representation is not unique. Indeed:

LEMMA 4 *For all natural numbers n and k ,*

$$(p-5)u_{n+k} + (p-5)u_n + \sum_{i=1}^{k-1} (p-6)u_{n+i} = u_{n+k+1} + u_{n-1} \quad (5)$$

Proof of Lemma 4. Using (3) we get:

$$\begin{aligned} (p-5)u_{n+k} + (p-6)u_{n+k-1} &= (p-4)u_{n+k} - u_{n+k} + (p-6)u_{n+k-1} \\ &= u_{n+k+1} - u_{n+k} + (p-5)u_{n+k-1} \end{aligned}$$

So, that:

$$\begin{aligned} (p-5)u_{n+k} + (p-6)u_{n+k-1} + (p-6)u_{n+k-2} \\ &= u_{n+k+1} - u_{n+k} + (p-5)u_{n+k-1} + (p-6)u_{n+k-2} \\ &= u_{n+k+1} - u_{n+k} + u_{n+k} - u_{n+k-1} + (p-5)u_{n+k-2} \\ &= u_{n+k+1} - u_{n+k-1} + (p-5)u_{n+k-2} \end{aligned}$$

By induction, on i we get that :

$$(p-5)u_{n+k} + \sum_{i=1}^{k-1} (p-6)u_{n+i} = u_{n+k+1} - u_{n+2} + (p-5)u_{n+1} \quad (*)$$

Adding $(p-5)u_n$ to both sides, we get:

$$\begin{aligned} (p-5)u_{n+k} + \sum_{i=1}^{k-1} (p-6)u_{n+i} + (p-5)u_n \\ &= u_{n+k+1} - u_{n+2} + (p-5)u_{n+1} + (p-5)u_n \\ &= u_{n+k-1} - ((p-4)u_{n+1} - u_n) + (p-5)u_{n+1} + (p-5)u_n \\ &= u_{n+k+1} - u_{n+1} + (p-4)u_n = u_{n+k+1} + u_{n-1} \end{aligned}$$

□

As $(p-5)u_n = (p-6)u_n + u_n$ we can rewrite (5) as:

$$(p-5)u_{n+k} + \left(\sum_{i=0}^{k-1} (p-6)u_{n+i} \right) + u_n = u_{n+k+1} + u_{n-1}$$

making $n = 0$ in the latter equation, as $u_{-1} = 0$ and $u_0 = 1$ and replacing k by n this provides us with

$$(p-5)u_n + \left(\sum_{i=0}^{n-1} (p-6)u_i \right) = u_{n+1} - 1 \quad (6)$$

Again, using that $u_0 = 1$ we can again write:

$$(p-5)u_n + \left(\sum_{i=1}^{n-1} (p-6)u_i \right) + (p-5)u_0 = u_{n+1} \quad (7)$$

From (7), replacing n by $n+k$ and taking a part under the sign \sum , we get:

$$(p-5)u_{n+k} + \sum_{i=1}^{k-1} (p-6)u_{n+i} < u_{n+k+1} \quad (8)$$

Note that (7) gives us another proof that the representation (4) is not unique. Together with (6), we obtain that we necessarily go to u_{n+1} just after the number given by the left-hand side of (6). We have : $[u_{n+1}-1] = \mathbf{b}_1 \mathbf{b}_2^n$. We can also say from (7) that the string $\mathbf{b}_1 \mathbf{b}_2^k \mathbf{b}_1$ is ruled out for any $k \in \mathbb{N}$.

Also note that the representation whose number of digits is greater is unique. It is obtained by the following algorithm:

```

INPUT:  $n$  and  $u_i$  for  $i < n$ , a table  $a$ ;
while  $u_i \leq n$  loop  $i := i+1$ ; end loop;
for  $j$  in reverse  $\{0..i-1\}$ 
loop  $a_j := n \text{ div } u_j$ ;  $n := n \bmod u_j$ ; end loop;
OUTPUT:  $a$  and  $i$ , the length of the table.

```

From these last remarks we get:

THEOREM 3 (Margenstern, see [4]) *The language of the coordinates for the tessellations $\{p-2, 4\}$ and $\{p, 3\}$ with $p \geq 7$ is rational.*

We have that $[(p-5)u_{n+k} + \sum_{i=1}^{k-1} (p-6)u_{n+i} + (p-5)u_n] = \mathbf{b}_1 \mathbf{b}_2^{k-1} \mathbf{b}_1 \mathbf{0}^n$, where

$\mathbf{b}_2 = p-6$ and that $[u_{n+k+1} + u_{n-1}] = \mathbf{10}^{k+1} \mathbf{10}^{n-1}$. The second representation has one more digits than the first one.

From now on, we take as coordinate of ν the representation $[\nu]$ with the greatest number of digits and we again denote it by $[\nu]$.

By induction, we define U_n by

$$U_0 = u_0 \text{ and } U_{n+1} = U_n + u_{n+1} \quad (9)$$

For our further study, we need a few results on the u_n 's and on the U_n 's.

LEMMA 5 *The sequences $\{u_n\}$ and $\{U_n\}$ are both increasing. For all positive n*

$$(p-5)u_n < u_{n+1} < (p-4)u_n \quad (10)$$

$$U_n + (p-6)u_n < u_{n+1} < U_n + (p-5)u_n \quad (11)$$

$$u_{n+1} < U_{n+1} - u_n \quad (12)$$

Proof of Lemma 5. From the equations (2) we derived from rules (1), we can see that the v_n 's and w_n 's are positive when $n > 0$. This proves that the sequences $\{v_n\}$ and $\{w_n\}$ are increasing which proves that the sequences $\{u_n\}$ and $\{U_n\}$ are also increasing. Summing the last two rows in (2) we get that $u_{n+1} = (p-4)w_n + (p-5)v_n$, whence the inequalities (10).

We prove the left-hand side inequality of (11) by induction:

$u_1 > U_0 + (p-6)u_0$ as $u_0 = U_0 = 1$ and as $u_1 = p-4$. Accordingly, assume that $u_{n+1} > U_n + (p-6)u_n$. Then, using the second equation of (3):

$$\begin{aligned} u_{n+2} &= (p-4)u_{n+1} - u_n = (p-6)u_{n+1} + u_{n+1} + u_{n+1} - u_n \\ &> U_n + (p-6)u_n + (p-6)u_{n+1} + u_{n+1} - u_n, \text{ by induction,} \\ &= U_n + u_{n+1} + (p-6)u_{n+1} + (p-7)u_n, \text{ so that, as } p \geq 7, \\ &\geq U_{n+1} + (p-6)u_{n+1}. \end{aligned}$$

The right-hand side of (11) is easier: as $u_n < U_n$ when $n \geq 1$,

$u_{n+1} < u_{n+1} + U_n - u_n < (p-4)u_n + U_n - u_n = U_n + (p-5)u_n$ from (10), so that we get the proof of (11). From the definition of U_{n+1} and from $u_n < U_n$ when $n \geq 1$ we easily get (12). \square

We need a similar lemma to Lemma 5 when, using the same rules as (1) we consider a tree \mathcal{T}_B whose root is a B -node. Denote by y_n the number of nodes on the level n and by Y_n the number of nodes in \mathcal{T}_B down to the level n , that one being included.

We have:

LEMMA 6 *The sequences $\{y_n\}$ and $\{Y_n\}$ are both increasing. For all positive n*

$$Y_{n+1} = Y_n + y_{n+1} \quad (13)$$

$$u_{n+1} = (p-5)u_n + y_n \quad (14)$$

$$y_{n+1} = u_{n+1} - u_n \quad (15)$$

$$Y_{n+1} = U_{n+1} - U_n \quad (16)$$

Proof of Lemma 6. Formula (13) comes from the definitions of Y_n and of y_n . As the rules (1) also apply to the nodes of \mathcal{T}_B formulas (2) also apply after replacing the first condition by $y_0 = 1$ and $y_1 = 4$. Formula (14) comes from the decomposition of a tree down to the level $n+1$ rooted at a W -node into $p-5$ copies of the same tree down to the level n and a copy of \mathcal{T}_B down to the level n thanks to the second rule of (1). Now, from (14) and from (3) we get (15). At last, from the definition of U_n and from (13) we get (16) by induction. \square

Now, we can transform (7) into:

$$u_{n+1} = U_n + 1 + (p-6)u_n + \sum_{k=0}^{n-1} (p-7)u_k \quad (17)$$

Indeed,

$$\begin{aligned}
U_n + 1 + (p-6)u_n + \sum_{k=0}^{n-1} (p-7)u_k &= U_n + 1 + (p-6)u_n + \sum_{k=0}^{n-1} (p-6)u_k - \sum_{k=0}^{n-1} u_k \\
&= U_n + 1 + (p-6)u_n + \sum_{k=0}^{n-1} (p-6)u_k - U_{n-1} = u_n + 1 + (p-6)u_n + \sum_{k=0}^{n-1} (p-6)u_k \\
&= (p-5)u_n + \sum_{k=1}^{n-1} (p-6)u_k + (p-5)u_0 = u_{n+1}
\end{aligned}$$

according to (7).

On another side, the definition of U_n allows us to write:

$$u_{n+1} = U_{n+1} - \sum_{k=0}^n u_k 1 \quad (18)$$

Formulas (17) and (18) will help us in the next Subsections.

3.2 The preferred son approach

We now turn to a closer study of the tree, forgetting for a while the connection with our tessellations. From this point, we shall denote a node either by its number ν or by its coordinate $[\nu]$.

The rules of (1) indicates that each node has a single B -node. The rules are a formal writing, they do not assign a place to the B -node among the sons of the node. We may decide a particular display. For a reason which will be later clear, let us consider this one:

$$\begin{aligned}
B &\rightarrow W^{p-7}BW \\
W &\rightarrow W^{p-6}BW
\end{aligned} \quad (19)$$

Denote by \mathcal{P}_W , \mathcal{P}_B the tree which is obtained by a recursive unlimited application of the rules (19) from a W -, B -node respectively. We call W -, B -**tree of height** n , the sub-tree of \mathcal{P}_W , \mathcal{P}_B respectively issued from the root, down to the level n , that level being included. Number the nodes of \mathcal{P}_W , starting from the root to which we give number 1, and then go down level after level and, on each level, from left to right. We identify a node with its number. To each node ν we associate the string $[\nu]$ which we call the **coordinate** of the node. Define the **signature** of a node to be the lowest digit of its coordinate. Define, for a node, its **son signature** as the string which displays the signatures of its sons, from left to right. We have the following property:

THEOREM 4 *In the tree \mathcal{P}_W , the son signature of any node is $2..b_\alpha 01$, with $p-4$, $p-5$ digits and $\alpha = 1, 2$ for a W -, B -node respectively. Consequently, the B -nodes are exactly those whose signature is 0 . In each node, a single son has a 0 signature: call it the **preferred son** of the node. In each node, the preferred son is the penultimate.*

Figure 4 shows the first two levels of \mathcal{P} . We can see that the statement of Theorem 4 is observed in the figure. We also call this tree the **preferred son tree**.

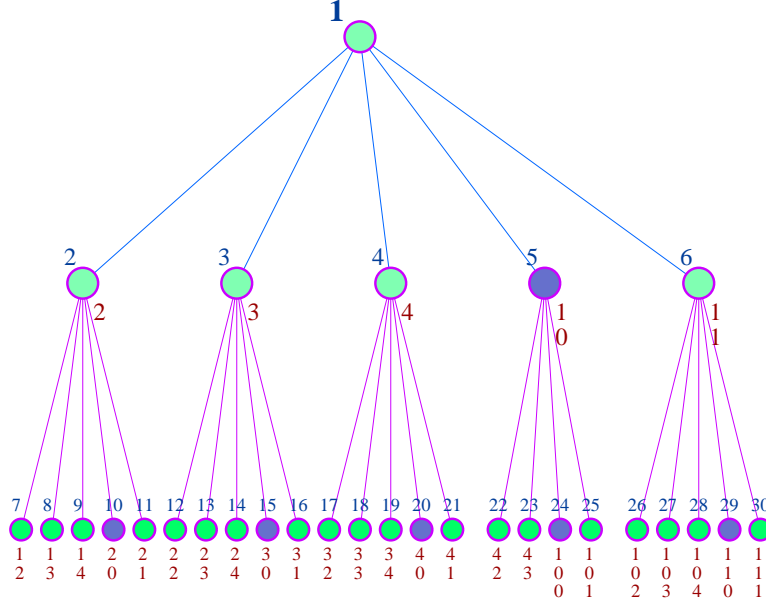


Figure 4 The tree associated to the rules (19). In that tree, the B -nodes are the preferred sons of the nodes. For each node: in blue, the numbers, in red the coordinates.

Proof of Theorem 4. Figure 4 shows us that the theorem is true for the root and for its sons, *i.e.* for level 1. As U_n is the number of nodes up to the level n , that level being included, U_n is also the number of the last node on the level, we also say the rightmost one, and its coordinate is $\mathbf{1}^{n+1}$. As a consequence, the coordinate of the first node of the level n is $\mathbf{1}^{n-1}\mathbf{2}$. Indeed, from (11) and (12), we get that $U_n < u_{n+1} < U_{n+1}$ so that $[\nu]$ has n digits when $\nu < u_n$ and $n+1$ digits when $\nu \geq u_n$. Next, from (11) and (12) again, we obtain that u_{n+1} always belong to the B -tree rooted at the B -son of the root. Formulas (17) and (18) allow us to precisely locate u_{n+1} in that tree. Indeed, consider formula (18) with $n = 0$. It says u_1 is the penultimate son of the root. If $n = 1$, it says that u_2 is the penultimate node of the B -tree rooted at u_1 . By induction, the formula says that u_{n+1} is the penultimate node of the B -tree rooted at u_n . Formula (17) says the same thing: instead of starting from U_{n+1} the last node on the level n , we start from U_n+1 , the first node on that level. The formula says that we cross the $p-6$ W -trees rooted at the first $p-6$ sons of the root of \mathcal{P} , and then we cross the $p-7$ W -trees of height $n-1$ rooted at the first $p-7$ W -sons of u_1 , then the $p-7$ W -trees of height $n-2$ rooted at the $p-7$ W -sons of u_2 and so on, until we arrive at u_n : we cross its $p-7$ W -sons before arriving to u_{n+1} . As the coordinate of u_{n+1} is $\mathbf{10}^n$ by definition, the signature of all u_k 's for $k \geq 1$ is $\mathbf{0}$ and, by the rules (19), they are B -nodes. Call the sequence of nodes $\{u_i\}_{i \in \mathbb{N}^+}$, where \mathbb{N}^+ denote the set of positive integers, the **main B -line** of the tree \mathcal{P}_W .

Fix a node ν in \mathcal{P}_W . Let \mathcal{P}_ν denote the sub-tree of \mathcal{P}_W rooted at ν : we have

that $\mathcal{P}_W = \mathcal{P}_1$. Call **B -thread** of ν the sequence of B -nodes $\{\nu_i\}_{i \in \mathbb{N}^+}$ such that ν_1 is the B -son of ν and ν_{i+1} is the B -son of ν_i for all $i \in \mathbb{N}$. Say that a node μ is in the left-, right-hand side of the B -thread if it is in a sub-tree of \mathcal{P}_ν rooted at one of the W -sons of ν which come before ν_1 , at the W -son of ν which comes after ν_1 respectively. We shall say that μ is on the left-, right-hand side of ν if it is on the left-right-hand side of its B -thread respectively.

The theorem is an easy corollary of the next lemma.

LEMMA 7 *Let $[\nu]\mathbf{a}$ be the coordinate of a node μ with $\mathbf{a} \in [0..\mathbf{b}_1]$. Then the coordinates of the sons of μ are given by the following table:*

j	$[\nu]\mathbf{j}_1\mathbf{2}$	$[\nu]\mathbf{j}_1\mathbf{3}$	$[\nu]\mathbf{j}_1\mathbf{i}$	$[\nu]\mathbf{j}_1\mathbf{b}_2$	$[\nu]\mathbf{j}_1\mathbf{b}_1$	$[\nu]\mathbf{j}\mathbf{0}$	$[\nu]\mathbf{j}\mathbf{1}$
0	$[\nu_1]\mathbf{b}_1\mathbf{2}$	$[\nu_1]\mathbf{b}_1\mathbf{3}$	$[\nu_1]\mathbf{b}_1\mathbf{i}$	$[\nu_1]\mathbf{b}_1\mathbf{b}_2$	$[\nu_1]\mathbf{0}$	$[\nu_1]\mathbf{1}$	

where $\nu_1 = \nu - 1$, $\mathbf{j} \in [1..\mathbf{b}_1]$ and $\mathbf{j}_1 = \mathbf{j} - 1$. Recall that $\mathbf{b}_2 = p - 6$. The first line gives the coordinates of the sons of μ when its signature is \mathbf{j} with $\mathbf{j} \neq \mathbf{0}$. The second line gives the coordinates when the signature of μ is $\mathbf{0}$.

Proof of Lemma 7. We proceed by induction on the level n of the node μ . We note that the lemma is true for the root and that it is also true for its sons as illustrated by Figure 4. Assume that the lemma is true up to the level n . Let μ be the first node of the level $n+1$. From what we know, its is $[\mathbf{1}^n\mathbf{2}]$ which we write $[\nu]\mathbf{2}$ with $\nu = [[\mathbf{1}^n]]$. The coordinate of the first son of μ is $[\mathbf{1}^{n+1}\mathbf{2}]$ which is of the form $[\nu]\mathbf{12}$. We can easily see that we also have that the coordinate of the i -th son of μ is $[\nu]\mathbf{1i}^1$ with $\mathbf{i}^1 = \mathbf{i} + 1$, for $i \leq p - 6$. Hence, for the $p - 6$ -th son, the coordinate is $[\nu]\mathbf{1b}_1$. For the next node, as b is not a digit, the coordinate is $[\nu]\mathbf{20}$, so that the coordinate of the last son is $[\nu]\mathbf{21}$ and for the first son of $\mu + 1$, the coordinate is $[\nu]\mathbf{22}$. Note that by its position among the sons of μ , its son $[\nu]\mathbf{20}$ is a B -node.

Now, we can repeat this argument for the sons of $\mu + 1$, $\mu + 2$, until $\mu + p - 6$. Accordingly, the coordinate of the first son of $\mu + p - 5$, which is a B -node is $[\nu]\mathbf{b}_1\mathbf{2}$. When we arrive to the $p - 7$ -th son of $\mu + p - 5$, its coordinate is $[\nu]\mathbf{b}_1\mathbf{b}_2$, so that for the next node, the B -son of $\mu + p - 5$, the coordinate is $[\nu + 1]\mathbf{00}$. From this, we can see that, arriving to the last son of the node $\mu + p - 4$, its coordinate is $[\nu + 1]\mathbf{11}$. This node is the last node on the tree rooted at the father of μ and of height 2. By induction of the height of the tree whose first node is μ , the induction being allowed by the position of the u_i 's given by (11) and (12), which coincide with a B -node, by the recursive application of the rules (19), we have that the rules apply to the tree \mathcal{P}_2 . We have proved the theorem for a tree of height n which is not rooted at 1. The similar argument still holds with \mathcal{P}_i with $i < p - 5$. Accordingly, we arrive at the first node on the level n of \mathcal{P}_5 which is a B -tree. Now, if we consider the sons of 5, namely nodes 22, 23, 24 and 25, see Figure 4. The first two nodes are W -nodes for which we can repeat the previous argument as \mathcal{P}_{22} and \mathcal{P}_{23} are trees with the height $n - 1$. The crossing at the level $n + 1$ of \mathcal{P}_W of \mathcal{P}_{24} which is a B -tree corresponds to a situation we have already met in the previous trees of height n and which is controlled by (11). We remain with \mathcal{P}_{25} which is again a W -tree, so that the theorem is also checked there, up to the considered level. So that we proved

the theorem for \mathcal{P}_5 . Now with \mathcal{P}_6 we have again a W -tree of height n , so that the theorem is also checked there. Consequently, we proved the theorem for the nodes on the level $n+1$. \square

An interesting consequence of the lemma is the following property:

THEOREM 5 *In each tree, the B -thread and the B -main line coincide. Define the \mathbf{a} -slice of a tree \mathcal{P}_ν as the nodes which are on the right-hand side of the \mathbf{a}_1 -th son of ν and on the left-hand side of its \mathbf{a} -th son when $\mathbf{a} \in [2..b_2]$, where $\mathbf{a}_1 = \mathbf{a} - 1$. The $\mathbf{0}$ -slice is on the right-hand side of the penultimate son and on the left-hand side of the last one, the $\mathbf{1}$ -slice is on the right-hand side of the last node and on the left-hand side of the first one, the \mathbf{b}_1 -slice, only present if ν is a W -node, is on the left-hand side of the penultimate node and on the right-hand side of the \mathbf{b}_2 -th node. Then, if $[\nu] = \mathbf{a}_k.. \mathbf{a}_1 \mathbf{a}_0$, then ν is in the \mathbf{a}_i -th slice of $[[\mathbf{a}_k.. \mathbf{a}_i]]$ when $i \in [1..k]$. We have that \mathbf{a}_0 gives the position in the tree rooted at $[[\mathbf{a}_k.. \mathbf{a}_1]]$.*

This provides us with an algorithm to locate the nodes, but we have to do more in order to get the branch, in the tree leading from 1 to the node. For this, we need a few easy lemmas:

LEMMA 8 *Let $\mathbf{a}_k.. \mathbf{a}_1 \mathbf{a}_0$ be the coordinate of a node ν . The coordinate of $\nu+1$ is given by Algorithm 1. The algorithm is linear in the size of $[\nu]$.*

ALGORITHM 1 *Algorithm for incrementing a number on its representation.*

```

i := 0;
while i ≤ k and  $\mathbf{a}_i = \mathbf{b}_2$ 
  loop i := i+1; end loop;
if i = 0 and  $\mathbf{a}_0 = \mathbf{b}_1$ 
  then  $\mathbf{a}_0 := 0$ ; k := k+1;  $\mathbf{a}_1 := 1$ ;
  elseif i = 0; -- then  $\mathbf{a}_0 < \mathbf{b}_2$ 
    then  $\mathbf{a}_0 := \mathbf{a}_0 + 1$ ;
  elseif i > k then  $\mathbf{a}_0 := \mathbf{a}_0 + 1$ ;
  elseif  $\mathbf{a}_i < \mathbf{b}_1$  then  $\mathbf{a}_0 := \mathbf{a}_0 + 1$ ;
  else for j in {0..i} loop  $\mathbf{a}_i := 0$ ; end loop;
    if i < k then  $\mathbf{a}_{i+1} := \mathbf{a}_{i+1} + 1$ ;
    else k := k+1;  $\mathbf{a}_k := 1$ ;
    end if;
  end if;

```

Proof of lemma 8. If $i = 0$ and $\mathbf{a}_0 \neq \mathbf{b}_2$, the body of the loop is never executed, so that after the **while**, nothing is changed and we must perform $\mathbf{a}_0 := \mathbf{a}_0 + 1$ unless $\mathbf{a}_0 = \mathbf{b}_1$, in which case we go from \mathbf{b}_1 to **10**. If we are not in this case, the body of the loop was executed at least once. The **while** achieves to perform the body of the loop, either because $i > k$ or because $\mathbf{a}_i = \mathbf{b}_2$. If

$i > k$, all digits are \mathbf{b}_2 , so that \mathbf{a}_0 must become \mathbf{b}_1 . The other possibility is that $i \leq k$ and so $\mathbf{a}_i \neq \mathbf{b}_2$. If $\mathbf{a}_i < \mathbf{b}_2$, we perform $\mathbf{a}_0 := \mathbf{a}_0 + 1$. If not, $\mathbf{a}_i = \mathbf{b}_1$ and so all \mathbf{a}_j 's from 0 to i must be $\mathbf{0}$ and \mathbf{a}_{i+1} becomes $\mathbf{a}_{i+1} + 1$. Indeed, as the pattern $\mathbf{b}_1\mathbf{b}_1$ is ruled out, we had $\mathbf{a}_{i+1} < \mathbf{b}_1$ before performing this action. Nevertheless, this assumes that $i < k$. If $i = k$, as $\mathbf{a}_k = \mathbf{b}_1$, k must be incremented by 1 and \mathbf{a}_{k+1} must be $\mathbf{1}$. \square

LEMMA 9 *Let $\mathbf{a}_k.. \mathbf{a}_1 \mathbf{a}_0$ be the coordinate of a node ν , assuming that $\nu \neq 0$. The coordinate of $\nu-1$ is given by Algorithm 2. The algorithm is linear in the size of $[\nu]$.*

ALGORITHM 2 *Algorithm for decrementing a number on its representation.*

```

i := 0;
while  $\mathbf{a}_i = \mathbf{0}$ 
loop  $\mathbf{a}_i := \mathbf{b}_2$ ;  $i := i+1$ ; end loop;
if  $i = 0$ 
    then  $\mathbf{a}_0 := \mathbf{a}_0 - 1$ ;
    else  $\mathbf{a}_i := \mathbf{a}_i - 1$ ;  $\mathbf{a}_{i-1} := \mathbf{b}_1$ ;
end if;

```

Proof of Lemma 9. Note that there is no need of a condition on i with respect to k in the **while** as $\mathbf{a}_k \neq \mathbf{0}$ according to the assumption that $\nu \neq 0$. If $i = 0$ after the **while**, the body of the loop was not executed, which means that $\mathbf{a}_0 \neq \mathbf{0}$, so that \mathbf{a}_0 becomes $\mathbf{a}_0 - 1$. If i is not 0 after the **while**, we arrive at \mathbf{a}_i which is not $\mathbf{0}$ and all \mathbf{a}_j 's with $j < i$ are transformed into \mathbf{b}_2 . We can reduce \mathbf{a}_i by 1 and, due to (6), as $i > 0$, we set \mathbf{a}_{i-1} to \mathbf{b}_1 . \square

At last, the following lemma reminds us how to recognize the status of a node and it allows also us to compute the coordinate of the father of a node. By convention, the father of the root is 0. The lemma gives both the number and the coordinate.

LEMMA 10 *Let ν be a node with $[\nu] = \mathbf{a}_k.. \mathbf{a}_1 \mathbf{a}_0$. It is a B-node if and only if $\mathbf{a}_0 = \mathbf{0}$. The father of ν is obtained as follows:*

If $\mathbf{a}_0 \in \{0, 1\}$, then the father is $[[\mathbf{a}_k.. \mathbf{a}_1]]$. Otherwise, it is $[[[\mathbf{a}_k.. \mathbf{a}_1]] + 1]$, where the latter number is computed by the algorithm of Lemma 9.

LEMMA 11 *Consider an \mathbf{a} -slice in \mathcal{P}_ν . Let ν_ℓ and ν_r be the sons of ν which delimit the slice. Then the digits of the sons of ν_ℓ and of ν_r which belong to the \mathbf{a} -slice are $[\mathbf{1}.. \mathbf{b}_1]$ if ν_r is a W-node. They belong to $[\mathbf{1}.. \mathbf{b}_2]$ if ν_r is a B-node.*

The lemma allows us to prove the correctness of Algorithm 3 stated in the next theorem. The algorithm constructs a path going from the root to the node, constituted by new digits in $\{1..p-4\}$. The first digit d_1 indicates the path from the root to the d_1 -th son of the root. If we arrived at a node ν through $d_1..d_i$,

d_{i+1} indicates the d_{i+1} -th node of ν . Clearly, $p-4$ is never used for a B -node and, the digit $p-5$ leads to the B -son in a W -node while in a B -node, the B -son is reached through the digit $p-6$.

ALGORITHM 3 *Algorithm to compute the path from the root to the node when given its coordinate.*

```

INPUT:  $\mathbf{a}_k.. \mathbf{a}_0$ ;  $\ell, r$ : tables of size  $k+1$ ;
       $prev := k$ ;  $s_\ell := W$ ,  $s_r := W$ ;
      procedure actualize ( $a, b, i, t$ ) is
      begin
        for  $j$  in  $[i..t]$ 
          loop  $a(j) := b(j)$ ; end loop;  $t := i$ ;
        end procedure;
for  $i$  in reverse  $[0..k]$ 
loop if  $\mathbf{a}_i$  in  $[2..b_1]$ 
  then  $\ell(i) := \mathbf{a}_i - 1$ ;  $r(i) := \mathbf{a}_i$ ;
    if  $((s_\ell = B) \text{ and } (\mathbf{a}_i = \mathbf{b}_2))$ 
      or  $((s_\ell = W) \text{ and } (\mathbf{a}_i = \mathbf{b}_1))$ 
    then  $s_\ell := W$ ;  $s_r := B$ ;
    else  $s_\ell := W$ ;  $s_r := W$ ;
    end if;
    actualize( $\ell, r, i+1, prev$ );
  elseif  $\mathbf{a}_i = 0$ 
    then if  $s_\ell := B$ ;
      then  $\ell(i) := p-6$ ;  $r(i) := p-5$ ;
      else  $\ell(i) := p-5$ ;  $r(i) := p-4$ ;
       $s_\ell := B$ ;  $s_r := W$ ;
    end if;
    actualize( $r, \ell, i+1, prev$ );
  elseif  $\mathbf{a}_i = 1$ 
    then if  $i = k$ 
      then  $\ell(i) := 0$ ;  $r(i) := 0$ ;
      else if  $s_\ell = B$ 
        then  $\ell(i) := p-5$ ;
        else  $\ell(i) := p-4$ ;
      end if;
       $r(i) := 1$ ;  $s_\ell := W$ ;  $s_r := W$ ;
    end if;
  end if;
end loop;
OUTPUT:  $\ell$ ;

```

THEOREM 6 (Margenstern, see [4]) *Algorithm 3 computes the branch from the root of \mathcal{P}_W to a node ν in $[\nu]$. The algorithm is linear in the length of $[\nu]$.*

Proof of Theorem 6. The idea of the algorithm is to use the property stated by Lemma 11. Each digit of the coordinate indicates in which slice of the tree the node occurs. Note that a slice is delimited by two nodes, say the **left-, right-hand side milestones**. The problem is to identify which milestone belongs to the path, in the tree, leading from the root to the node.

We construct the path by using two tables, ℓ and r : ℓ , r contains the path from the root to the left-, right-hand side milestone of the current slice respectively. As indicated before, the path consists in directions encoded by digits in $\{1..p-4\}$. Assume that the paths are ℓ and r for the left-, right-hand side milestone of the current slice respectively. Let \mathbf{a} be the new digit.

If $\mathbf{a} \in \{2..b_1\}$, then the left-hand side milestone is a son of the right-hand side one at the previous step. So it is also the case for the new right-hand side milestone. Eventually, we have to copy r onto ℓ from the last index until which the contents of tables were equal. If $\mathbf{a} = \mathbf{0}$, then, the left-hand side milestone is the B -son of the previous left-hand side milestone ℓ_0 , and the right-hand side milestone is the last son of ℓ_0 . We have to eventually copy a part of ℓ onto r . If $\mathbf{a} = \mathbf{1}$, then the new left-hand side milestone is the last son of the previous one while the new right-hand side milestone is the first son of the previous one. In that case, there is no actualization but there will be later. Note that the node whose coordinate is defined by the digits already visited is given by the path defined by the content of ℓ down to this digit. This leads us to Algorithm 3. This completes the proof of Theorem 6. \square

3.3 The leftmost approach

The leftmost approach is based on the decomposition we have seen in the illustrations given by Figures 1 and 3 and in the process described in the proofs of Theorems 1 and 2. It attaches the B -node to the strip so that if the orientation from left to right is identified with a counter-clockwise motion around a point in the hyperbolic plane, then the B -node is the leftmost son of a node, whence the title of the current section. We call this new tree the **leftmost son tree**.

The numbering of the nodes is the same as the rules define the same number of B -sons and W -sons as in the Subsection 3.2. We also keep the same definition of the coordinates. The new tree is not exactly the same as the previous one: the branches are different, except in some part of the tree as we shall see.

Consider the representation of the tree given in Figure 5. We can check on the figure that the root is a W -node and that each node has one B -son exactly. Number the nodes starting from 1 which we assign to the root and then, level after level and on each level, from left to right.

In the figure, we also can remark the following property. The signature of some nodes is 0, and it seems that each node has one son exactly whose signature is 0. This property is true for each node of the tree as below stated. As the branches of the tree are different from that of Subsection 3.2, this requires a proof.

THEOREM 7 Consider the leftmost son tree \mathcal{T} , which is associated to a sector of the tessellations $\{p, 3\}$ and $\{p-2, 4\}$ with $p \geq 7$. The tree has two kinds of nodes, B- and W-nodes which are distributed in the tree according to the rules 1. Among the sons of the node n , exactly one has $[n]0$ as its coordinate. Call it again the **preferred** son of the node n . In the B-nodes, the preferred son is the rightmost one. Consider a W-node n . Say that its **type** is 1 if the preferred son is the rightmost one, otherwise, say that it is 2. In any node, the preferred son is always a W-node of type 2. The root of the tree is a W-node of type 2. In a W-node of type 1, all W-sons but the preferred one are of type 1. In a W-node of type 2, all W-sons but two of them are of type 1. The two W-sons of type 2 in a node of type 2 are its preferred son and its rightmost son.

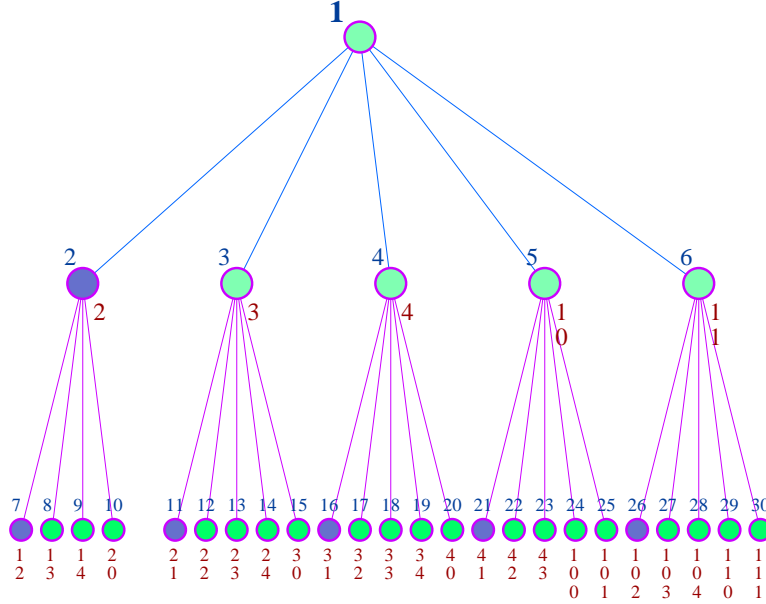


Figure 5 The associated with the decomposition of a sector of the tessellations as indicated in Figures 1 and 2. In that tree, the B-nodes are the leftmost son of a node. Here too, for each node, the number appears in blue, above the node, the coordinate is written in red, below, vertically.

Proof of Theorem 7. As the nodes with signature $\mathbf{0}$ are the same in both trees, we prove the theorem by comparing the distributions of the sub-trees in each tree. As the number of nodes are the same, we remark that the differences of distribution appear inside the W-trees.

Let us compare the first level of the trees in Figures 4 and 5. It appears that the difference is a permutation operating on the first and the $p-5$ -th sons of the root which is a W-node in both cases. As a B-node has one node less than a W-node, the above permutation entails a shift on the position of the trees issued from the nodes of the first level compared to the similar trees in the case of the preferred son display. Accordingly, up to u_2 , the preferred son of the nodes 2 up

to $p-5$ being included is the rightmost son. Now, as the position of the leftmost and rightmost branches of the tree issued from the last node, which is a W -node in both trees, in that tree the preferred son is the penultimate son, as it is in the preferred son tree. We can see that there are two kinds of W -nodes: those we call of **type 1**, where the preferred son is the last one, and those of **type 2** where the preferred son is the penultimate one. We note that the root is a W -node of type 2. Its last son is also of type 2. Now, its preferred son ω is now a W -node. Let \mathcal{W} be W -tree issued from ω in the leftmost son tree and let \mathcal{B} be the B -tree issued from ω in the preferred son tree. From what we noticed on the trees issued from the last node of the root, \mathcal{W} and \mathcal{B} have the same rightmost branch. On the first level of \mathcal{W} compared to that of \mathcal{B} , the additional node is outside \mathcal{B} . In particular, as the preferred sons are the same in the preferred son and the leftmost son trees, the preferred son of ω has the same position with respect to the rightmost branch of both \mathcal{B} and \mathcal{W} . Accordingly, \mathcal{W} is also of type 2. We also note that the leftmost and the preferred displays inside a B -tree shows us the same permutation between the B -sons, namely between the first one and the penultimate one. Consequently, what we notice for B - and W -sons also holds for leftmost son B -trees. The theorem is proved. \square

In this context, we have a property which is analogous to that which is stated in Theorem 6.

THEOREM 8 (Margenstern, see [4]) *Algorithm 4 computes the branch from the root of \mathcal{P}_W to a node ν of a leftmost son tree in $[\nu]$. The algorithm is linear in the length of $[\nu]$.*

LEMMA 12 *The son signatures of the nodes in a leftmost son tree are the following ones:*

$$\begin{array}{ll}
B\text{-node} & \mathbf{2..b_10} \\
W\text{-node of type 1} & \mathbf{1..b_10} \\
W\text{-node of type 2} & \mathbf{2..b_101} \\
& \mathbf{1..b_201}
\end{array} \tag{20}$$

Proof of Lemma 12. The proof comes from the fact that the son signature is always $\mathbf{2..b_101}$ in preferred son trees and on the shift we observed in the proof of Theorem 7. By induction, assume that the lemma is true for the node ν . If ν is a B -node, its signature is $\mathbf{2..b_10}$. Now, the node $\nu+1$ is necessarily a W -node of type 1 and in its signature, the first digit is $\mathbf{1}$. As a W -node has $p-4$ nodes, the signature is $\mathbf{1..b_10}$. If ν is a W -node of type 1, its signature is $\mathbf{1..b_10}$. Hence, the first digit of the signature of $\nu+1$ is $\mathbf{1}$. As $\nu+1$ is either a W -node of type 1 or a W -node of type 2, its signature is $\mathbf{1..b_10}$ or $\mathbf{1..b_201}$ respectively: in a W -node of type 2, the preferred son is the penultimate. If ν is a W -node of type 2, the last digit of its signature is $\mathbf{1}$ as the preferred son is the penultimate. Hence, whatever $\nu+1$, either a W -node of type 2 or a B -node, the first digit of its signature is $\mathbf{2}$. Hence If $\nu+1$ is a W -node of type 2, a B -node, its signature is $\mathbf{2..b_201}$, $\mathbf{2..b_20}$ respectively. Accordingly, the signature of $\nu+1$ is one of the signatures indicated in the lemma according to the status of the node. \square

ALGORITHM 4 *Algorithm for computing the path from the root to a node in the leftmost son tree.*

```

INPUT:  $\mathbf{a}_k.. \mathbf{a}_0$ ;  $\ell, r$ : tables of size  $k+1$ ;
       $prev := k$ ;
       $s_\ell := 0$ ;  $s_r := 0$ ;
      procedure actualize ( $a, b, i, t$ ) is
      begin
        for  $j$  in  $[i..t]$ 
          loop  $a(j) := b(j)$ ; end loop;  $t := i$ ;
        end procedure;
for  $i$  in reverse  $[0..k]$ 
loop if  $s_\ell = 0$  and  $s_r = 0$  and  $i = k$ 
  then if  $\mathbf{a}_i = 1$ 
    then  $s_\ell := W_2$ ;  $s_r := B$ ;  $\ell(i) := 0$ ;  $r(i) := 0$ ;
    else  $s_\ell := W_1$ ;  $s_r := W_1$ ;  $\ell(i) := \mathbf{a}_i - 1$ ;  $r(i) := \mathbf{a}_i$ ;
      if  $\mathbf{a}_i = \mathbf{b}_1$  then  $s_r := W_2$ ; end if;
    end if;
  elseif ( $s_\ell := B$  and  $s_r := W_1$ ) or ( $s_\ell := W_1$  and  $s_r = W_1$ )
    or ( $s_\ell = W_1$  and  $s_r = W_2$ )
    then if  $\mathbf{a}_i$  in  $[1..b_1]$ 
      then  $\ell(i) := \mathbf{a}_i$ ;  $r(i) := \mathbf{a}_i + 1$ ;  $s_\ell := W_1$ ;
        if  $\mathbf{a}_i = \mathbf{b}_2$  and  $s_r = W_2$  then  $s_r := W_2$ ;
          elseif  $\mathbf{a}_i < \mathbf{b}_1$  then  $s_r := W_1$ ; else  $s_r := W_2$ ;
        end if;
        actualize( $\ell, r, i+1, prev$ );
      else  $r(i) := 1$ ;  $\ell(i) := p-5$ ;
        if  $s_\ell = W_1$  then  $\ell(i) := \ell(i) + 1$ ; end if;
         $s_\ell := W_2$ ;  $s_r := B$ ;
      end if;
    elseif ( $s_\ell := W_2$  and  $s_r := W_2$ ) or ( $s_\ell := W_2$  and  $s_r = B$ )
      then if  $\mathbf{a}_i$  in  $[2..b_1]$ 
        then  $\ell(i) := \mathbf{a}_i - 1$ ;  $r(i) := \mathbf{a}_i$ ;  $s_\ell := W_1$ ;
          if  $\mathbf{a}_i < \mathbf{b}_1$  then  $s_r := W_1$ ; else  $s_r := W_2$ ; end if;
          actualize( $\ell, r, i+1, prev$ );
        elseif  $\mathbf{a}_i = 0$  then  $\ell(i) := p-5$ ;  $r(i) := p-4$ ;
           $s_\ell := W_2$ ;  $s_r := W_2$ ;
          actualize( $r, \ell, i+1, prev$ );
        else  $\ell(i) := p-4$ ;  $r(i) := 1$ ;  $s_\ell := W_2$ ;  $s_r := B$ ;
      end if;
    end if;
  end loop;
OUTPUT:  $\ell$ ;

```

To prove the theorem, we need the following property:

It is interesting to pay a new visit to Theorem 6 in the context of the leftmost son tree. The theorem is still true in this new frame. We again state it in this new context as the tree being different, the algorithm is also different.

Proof of Theorem 8. The basic point of the proof, namely Theorem 5, is true. Indeed, the proof of that theorem lies on the notion of slice. As here B -nodes and node with signature $\mathbf{0}$ always being different, we replace the B -nodes by the preferred son: the **posterity** of a node ν is a sequence of nodes $\{\nu_i\}_{i \in \mathbb{N}}$ such that $\nu_0 = \nu$ and ν_{i+1} is the preferred son of ν_i . In the tree rooted at ν , the slices are again the set of nodes between the posterities of two consecutive sons of ν . Now, from this definition, we can see that the slices are the same in a preferred son tree and in a leftmost son one. The differences between the trees are the branches in the slices. Now, what is changed with respect to the situation of Theorem 6 is the delimitations of the sub-trees. We have six possible situations, depending on the values of s_ℓ and s_r , the type of node of the current node. They are indicated by the following pairs:

$$0 - 0 \quad B - W_1 \quad W_1 - W_1 \quad W_1 - W_2 \quad W_2 - W_2 \quad W_2 - B$$

Note that in the situation $W_1 - W_2$, the son signature of the W_2 -node is **12..b₂01**, that in the situation $W_2 - W_2$, the son signature of the first W_2 -node is **12..b₂01** and that of the second one is **2..b₁01**. At last, in the situation $W_2 - B$, the son signature of the W_2 -node is **2..b₁01**. Note that $0 - 0$ is the situation of the beginning of the process.

As in the case of the preferred son tree, we construct two paths ℓ and r . The current node, corresponding to the already examined digits, is always reached by ℓ . This leads us to Algorithm 4. \square

4 The coordinates of a node and of its neighbours

The preferred son and the leftmost trees are different but, as they both are sub-graphs of the dual graph of the tiling, each node has the same neighbours, whichever the tree.

It is interesting to restore the dual graph from the tree. As the leftmost son tree is tightly connected with the decomposition we introduced in Section 2 in both tilings, it is not difficult to see that, in the case of the tessellations $\{p-2, 4\}$, $\{p, 3\}$, the $p-2$, $p-3$ -th respectively neighbour of a tile ν is the first son of the tile $\nu+1$. This can easily be performed in the leftmost son tree: to each node, we append a connection between the tile ν and the first son of $\nu+1$ which is the next node after the last son of ν on the level of its sons. For the case of the tessellations $\{p, 3\}$ there are two other connections for ν : the connections with its neighbours on the same level, namely the nodes $\nu-1$ and $\nu+1$. As the nodes are the same in both trees as well as their dual graph, we can see that the connections are a bit more complex to be established in the case of the preferred son tree.

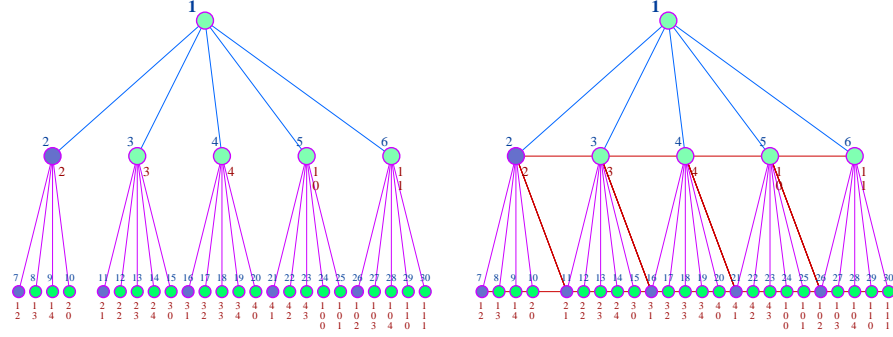


Figure 6 To left, the leftmost son tree. To right: the dual graph restored from the leftmost son tree. The dual graph for the tessellation $\{p-2, 4\}$ is obtained by removing the horizontal red arcs.

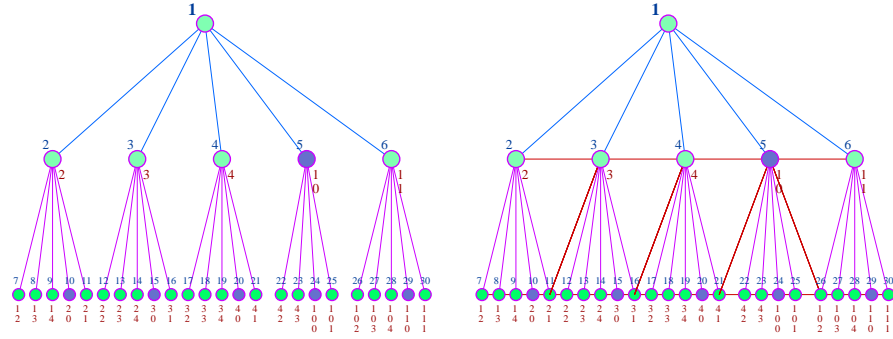


Figure 7 To left, preferred son tree. To right: the dual graph restored from the preferred son tree. The dual graph for the tessellation $\{p-2, 4\}$ is obtained by removing the horizontal red edges.

Consider the level 2 in both the leftmost son and the preferred son trees, \mathcal{T} and \mathcal{P} respectively. We know the permutation which allows to pass from one tree to the other one and the shift by one node which it entails in between the positions on which the permutation operates. Figures 6, 7 illustrate the transformation of \mathcal{T} , \mathcal{P} respectively into the dual graph. Figure 8 displays both restored graphs: it allows us to compare the processes and to define the transformation from the tree to the dual graph in the case of the preferred son tree.

On Figure 8, we can see that the situation is different depending on which node we consider: the nodes which are the sons of a given node ν and which occur before the preferred son are applied a rule which is different from the rule which is applied to the preferred son.

In \mathcal{T} , in order to restore the dual graph, we append from each node the horizontal arcs which connect the node to its neighbours belonging to the same level. Outside these arcs, we append an additional one to the right: from the node to the first son of its right-hand side neighbour.

In \mathcal{P} we do the same for the horizontal arcs. The additional arc is appended in a different way. Consider a node ν whose father is μ , the level of ν being n . If ν is a W -node which is not the rightmost son of μ , its additional arc connects it with the rightmost son of $\nu-1$. If ν is a B -node, it has two additional arcs, one to the left, connected to the rightmost son of $\nu-1$, and one to the right, connected to the leftmost son of $\nu+1$. Hence, if ν is the rightmost son of μ , its additional arc is connected with $\mu-1$. This is illustrated by Figures 7 and 8.

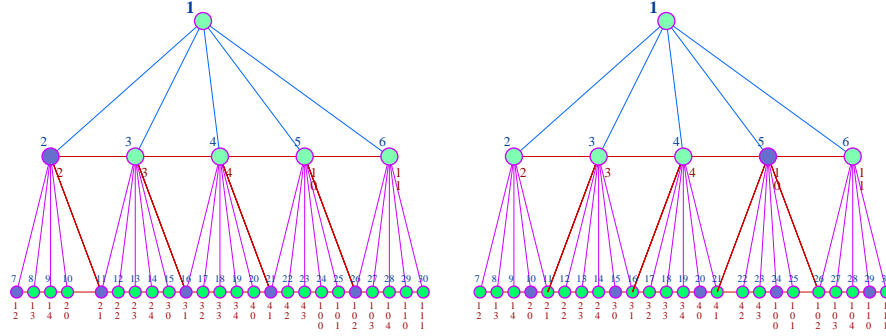


Figure 8 Comparison of the trees for restoring the dual graph. To left, the dual graph restored from the leftmost son tree. To right: the dual graph restored from the preferred son tree.

We are now in the position to define the coordinates of the neighbours of a node in both trees, first for the tessellations $\{p, 3\}$.

Consider a node ν in \mathcal{T} . We denote by $f(\nu)$ the father of ν and by $\sigma(n)$ its preferred son. The coordinates are given in Table 1 for \mathcal{T} and \mathcal{P} . The table also mention the signature of the sons of the nodes in each tree.

For the leftmost son tree \mathcal{T} , we indicate the W_1 -nodes, the two possible forms of the W_2 -nodes and the B -node. For the preferred son tree \mathcal{P} we indicate the three possible W -nodes: W_β for the leftmost son of its father, W_ℓ for the other W -nodes which are to the left of the B -son of their father, W_r for the rightmost son of its father.

A particular mention is given to the nodes which are on an extremal branch of the tree: the connections are the same for both types of tree: the nodes of the leftmost branch receive an arc from the nodes of the rightmost branch of the previous tree from the previous level. This fixes the rule for the nodes of the rightmost branch. Table 2 gives the coordinates in the case of the tessellations $\{p-2, 4\}$: it is enough to cancel the horizontal connections.

Table 1 *The neighbours of a node for the tessellation $\{p, 3\}$.*

leftmost son tree								
	W_1		W_2		W_2		B	
0	ν		ν		ν		ν	
1	$f(\nu)$		$f(\nu)$		$f(\nu)$		$f(\nu)$	
2	$\nu-1$		$\nu-1$		$\nu-1$		$f(\nu)-1^*$	
3	$\sigma(\nu)-p+5$	1	$\sigma(\nu)-p+6$	1	$\sigma(\nu)-p+6$	2	$\nu-1^*$	
4	$\sigma(\nu)-p+6$	2	$\sigma(\nu)-p+7$	2	$\sigma(\nu)-p+7$	3	$\sigma(\nu)-p+6$	2
$p-6$	$\sigma(\nu)-4$		$\sigma(\nu)-3$		$\sigma(\nu)-3$		$\sigma(\nu)-4$	
$p-5$	$\sigma(\nu)-3$		$\sigma(\nu)-2$		$\sigma(\nu)-2$	b₂	$\sigma(\nu)-3$	
$p-4$	$\sigma(\nu)-2$	b₂	$\sigma(\nu)-1$	b₂	$\sigma(\nu)-1$	b₁	$\sigma(\nu)-2$	b₂
$p-3$	$\sigma(\nu)-1$	b₁	$\sigma(\nu)$	0	$\sigma(\nu)$	0	$\sigma(\nu)-1$	b₁
$p-2$	$\sigma(\nu)$	0	$\sigma(\nu)+1$	1	$\sigma(\nu)+1$	1	$\sigma(\nu)$	0
$p-1$	$\sigma(\nu)+1$		$\sigma(\nu)+2$		$\sigma(\nu)+2^*$		$\sigma(\nu)+1$	
p	$\nu+1$		$\nu+1$		$\nu+1^*$		$\nu+1$	

preferred son tree								
	W_β		W_ℓ		W_r		B	
0	ν		ν		ν		ν	
1	$f(\nu)$		$f(\nu)$		$f(\nu)$		$f(\nu)$	
2	$f(\nu)-1^*$		$\nu-1$		$\nu-1$		$\nu-1$	
3	$\nu-1^*$		$\sigma(\nu)-p+5$		$\sigma(\nu)-p+6$	2	$\sigma(\nu)-p+6$	
4	$\sigma(\nu)-p+6$	2	$\sigma(\nu)-p+6$	2	$\sigma(\nu)-p+7$	3	$\sigma(\nu)-p+7$	2
$p-6$	$\sigma(\nu)-4$		$\sigma(\nu)-4$		$\sigma(\nu)-3$		$\sigma(\nu)-3$	
$p-5$	$\sigma(\nu)-3$		$\sigma(\nu)-3$		$\sigma(\nu)-2$	b₂	$\sigma(\nu)-2$	
$p-4$	$\sigma(\nu)-2$	b₂	$\sigma(\nu)-2$	b₂	$\sigma(\nu)-1$	b₁	$\sigma(\nu)-1$	b₂
$p-3$	$\sigma(\nu)-1$	b₁	$\sigma(\nu)-1$	b₁	$\sigma(\nu)$	0	$\sigma(\nu)$	0
$p-2$	$\sigma(\nu)$	0	$\sigma(\nu)$	0	$\sigma(\nu)+1$	1	$\sigma(\nu)+1$	1
$p-1$	$\sigma(\nu)+1$	1	$\sigma(\nu)+1$	1	$\sigma(\nu)+2^*$		$\sigma(\nu)+2$	
p	$\nu+1$		$\nu+1$		$\nu+1^*$		$\nu+1$	

*: The neighbours are different if the node is on an extremal branch. They belong to another tree: the previous one for B , W_β the next one for W_2 , W_r .
For B and W_β , neighbour 2: $\nu-1$; neighbour 3: $\sigma(\nu-1)+1$.
For W_2 and W_r : neighbour $p-1$: $\nu+1$; neighbour p : $f(\nu)+1$.

Table 2 The neighbours of a node for the tessellation $\{p-2, 4\}$.

leftmost son tree								
	W_1		W_2		W_2		B	
0	ν		ν		ν		ν	
1	$f(\nu)$		$f(\nu)$		$f(\nu)$		$f(\nu)$	
2	$\sigma(\nu)-p+5$	1	$\sigma(\nu)-p+6$	1	$\sigma(\nu)-p+6$	2	$f(\nu)-1^*$	
3	$\sigma(\nu)-p+6$	2	$\sigma(\nu)-p+7$	2	$\sigma(\nu)-p+7$	3	$\sigma(\nu)-p+6$	2
$p-6$	$\sigma(\nu)-3$		$\sigma(\nu)-2$		$\sigma(\nu)-3$	b₂	$\sigma(\nu)-3$	
$p-5$	$\sigma(\nu)-2$	b₂	$\sigma(\nu)-1$	b₂	$\sigma(\nu)-2$	b₁	$\sigma(\nu)-2$	b₂
$p-4$	$\sigma(\nu)-1$	b₁	$\sigma(\nu)$	0	$\sigma(\nu)-1$	0	$\sigma(\nu)-1$	b₁
$p-3$	$\sigma(\nu)$	0	$\sigma(\nu)+1$	1	$\sigma(\nu)+1$	1	$\sigma(\nu)$	0
$p-2$	$\sigma(\nu)+1$		$\sigma(\nu)+2$		$\sigma(\nu)+2^*$		$\sigma(\nu)+1$	
preferred son tree								
	W_β		W_ℓ		W_r		B	
0	ν		ν		ν		ν	
1	$f(\nu)$		$f(\nu)$		$f(\nu)$		$f(\nu)$	
2	$f(\nu)-1^*$		$\sigma(\nu)-p+5$		$\sigma(\nu)-p+6$	2	$\sigma(\nu)-p+6$	
3	$\sigma(\nu)-p+6$	2	$\sigma(\nu)-p+6$	2	$\sigma(\nu)-p+7$	3	$\sigma(\nu)-p+7$	2
$p-6$	$\sigma(\nu)-3$		$\sigma(\nu)-3$		$\sigma(\nu)-2$	b₂	$\sigma(\nu)-2$	
$p-5$	$\sigma(\nu)-2$	b₂	$\sigma(\nu)-2$	b₂	$\sigma(\nu)-1$	b₁	$\sigma(\nu)-1$	b₂
$p-4$	$\sigma(\nu)-1$	b₁	$\sigma(\nu)-1$	b₁	$\sigma(\nu)$	0	$\sigma(\nu)$	0
$p-3$	$\sigma(\nu)$	0	$\sigma(\nu)$	0	$\sigma(\nu)+1$	1	$\sigma(\nu)+1$	1
$p-2$	$\sigma(\nu)+1$	1	$\sigma(\nu)+1$	1	$\sigma(\nu)+2^*$		$\sigma(\nu)+2$	

*: The neighbours are different if the node is on an extremal branch. They belong to another tree: the previous one for B , W_β the next one for W_2 , W_r .

For B and W_β , neighbour 2: $\nu-1$.

For W_2 and W_r : neighbour $p-2$: $\nu+1$.

We conclude this section by a visit to the tree which is common to the pentagrid, the tessellation $\{5, 4\}$, and to the heptagrid, the tessellation $\{7, 3\}$. Figure 9 displays the trees associated to the leftmost son representation and the preferred son one.

In the preferred son tree, we can see the following properties:

THEOREM 9 *When $p = 7$ the preferred son tree has the following properties. The tree has two kinds of nodes, W - and B -nodes. In B -nodes, the preferred son is the leftmost one, in the W -nodes, it is the penultimate. The son signatures*

are the following ones:

$$\begin{array}{cc} B & W \\ \mathbf{0} & \mathbf{1} \end{array} \quad \begin{array}{cc} W \\ \mathbf{2} & \mathbf{0} & \mathbf{1} \end{array} \quad (21)$$

Proof of Theorem 9. The polynomial we obtain from the decomposition of a sector is this time:

$$P(X) = X^2 - 3X + 1 \quad (22)$$

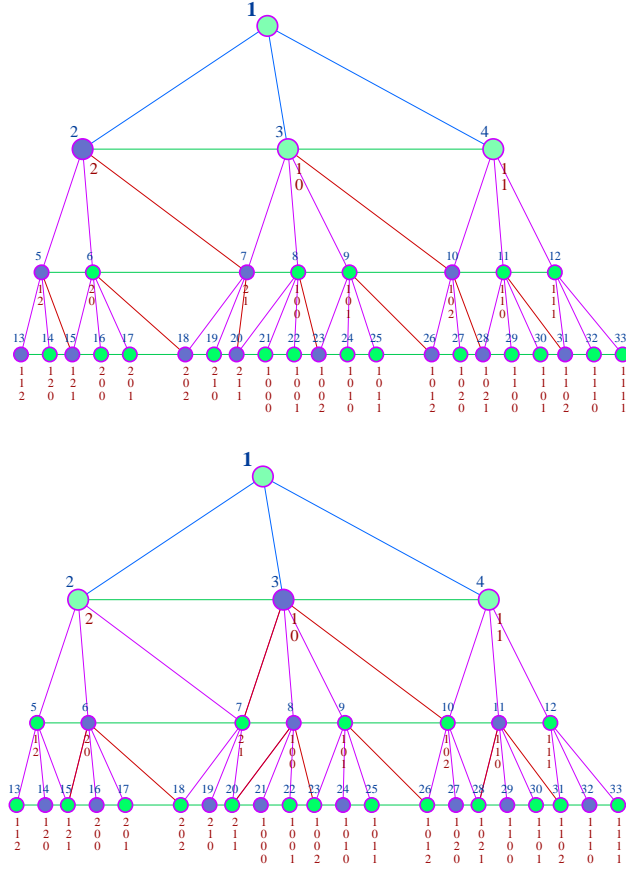


Figure 9 To left, the leftmost son tree. To right: the dual graph restored from the leftmost son tree. The dual graph for the tessellation $\{p-2, 4\}$ is obtained by removing the horizontal red edges.

This corresponds to the value $p = 7$ in the polynomial obtained from the equations (3). From this, it is easy to see that the equation (17) becomes:

$$u_{n+1} = U_n + u_n + 1 \quad (23)$$

This tells that the first node on the level $n+1$ we wind after we crossed the sub-tree rooted at the node 2 is the node u_{n+1} . It also confirms that the

sequence of nodes $\{u_n\}_{n \in \mathbb{N}^*}$ satisfies the property that u_{n+1} is a B -node which is the preferred son of the node u_n which is also a B -node for $n > 0$, by definition. Now, formula (6) can be rewritten as:

$$2u_n + \left(\sum_{i=0}^{n-1} u_i \right) = u_{n+1} - 1 \quad (24)$$

telling us that $[u_{n+1}-1] = \mathbf{21}^n$, which also says that the pattern $\mathbf{21}^* \mathbf{1}$ is ruled out in the writing of a coordinate. From these observations the theorem easily follows. \square

Turning now to the leftmost son tree, there we can see the following properties:

THEOREM 10 *When $p = 7$ the leftmost son tree has the following properties. The tree has two kinds of nodes, W - and B -nodes. In B -nodes, the preferred son is the rightmost one, in the W -nodes, it is the penultimate. The son signatures are the following ones:*

$$\begin{array}{ccc} B & W_0 & W_1 \\ \mathbf{2} \mathbf{0} & \mathbf{1} \mathbf{0} \mathbf{1} & \mathbf{2} \mathbf{0} \mathbf{1} \end{array} \quad (25)$$

where W_0 is a W -node whose signature is $\mathbf{0}$ and W_1 is a W -node whose signature is $\mathbf{1}$. Digit $\mathbf{2}$ is always the signature of a B -node.

Proof of Theorem 10. The theorem is a corollary of Theorem 9 as we can apply to this tree and recursively to its sub-trees the permutation between the first two nodes in each W -tree. Accordingly, the son signatures of the first level which are the first line below become what the second line indicates.

$$\begin{array}{l} \mathbf{2} \mathbf{0} \mathbf{1} \text{ -- } \mathbf{0} \mathbf{1} \text{ -- } \mathbf{2} \mathbf{0} \mathbf{1} \text{ -- } \mathbf{0} \mathbf{1} \text{ -- } \mathbf{2} \mathbf{0} \mathbf{1} \text{ -- } \mathbf{2} \mathbf{0} \mathbf{1} \text{ -- } \mathbf{0} \mathbf{1} \text{ -- } \mathbf{2} \mathbf{0} \mathbf{1} \\ \mathbf{2} \mathbf{0} \text{ -- } \mathbf{1} \mathbf{0} \mathbf{1} \text{ -- } \mathbf{2} \mathbf{0} \text{ -- } \mathbf{1} \mathbf{0} \mathbf{1} \text{ -- } \mathbf{2} \mathbf{0} \mathbf{1} \text{ -- } \mathbf{2} \mathbf{0} \text{ -- } \mathbf{1} \mathbf{0} \mathbf{1} \text{ -- } \mathbf{2} \mathbf{0} \mathbf{1} \end{array}$$

This distribution is repeated in each sub-tree from one generation to another one. \square

Table 3 *The neighbours of a node in the heptagrid, the tessellation $\{7, 3\}$ in both trees.*

	preferred son tree			leftmost son tree		
	B	W_b	W_r	B	W_0	W_1
0	ν	0	ν	ν	ν	0
1	$f(\nu)$	$f(\nu)$	$f(\nu)$	$f(\nu)$	$f(\nu)$	$f(\nu)$
2	$\nu-1$	$\nu-1^*$	$\nu-1$	$f(\nu)-1^*$	$\nu-1$	$\nu-1$
3	$\sigma(\nu)-1$	$\sigma(\nu)-1^*$ 2	$\sigma(\nu)-1$ 2	$\nu-1^*$	$\sigma(\nu)-1$ 1	$\sigma(\nu)-1$ 2
4	$\sigma(\nu)$ 0	$\sigma(\nu)$ 0	$\sigma(\nu)$ 0	$\sigma(\nu)-1$ 2	$\sigma(\nu)$ 0	$\sigma(\nu)$ 0
5	$\sigma(\nu)+1$ 1	$\sigma(\nu)+1$ 1	$\sigma(\nu)+1$ 1	$\sigma(\nu)$ 0	$\sigma(\nu)+1$ 1	$\sigma(\nu)+1$ 1
6	$\sigma(\nu)+2$	$\nu+1$	$\sigma(\nu)+2^*$	$\sigma(\nu)+1$	$\sigma(\nu)+2$	$\sigma(\nu)+2^*$
7	$\nu+1$	$f(\nu)+1$	$\nu+1^*$	$\nu+1$	$\nu+1$	$\nu+1^*$

As in Table 1, with $*$ we indicate that the neighbours are different if the

node is on an extremal branch. They belong to another tree: the previous one for B , W_b the next one for W_1 , W_r .

For B and W_b , neighbour 2: $\nu-1$; neighbour 3: $\sigma(\nu-1)+1$.

For W_1 and W_r : neighbour 6: $\nu+1$; neighbour 7: $f(\nu)+1$.

Table 4 *The neighbours of a node in the pentagrid, the tessellation $\{5,4\}$ in both trees.*

	preferred son tree			leftmost son tree		
	B	W_b	W_r	B	W_0	W_1
0	ν	0	ν	ν	ν	0
1	$f(\nu)$	$f(\nu)$	$f(\nu)$	$f(\nu)$	$f(\nu)$	$f(\nu)$
2	$\sigma(\nu)-1$	$\sigma(\nu)-1^*$	$\sigma(\nu)-1$	$f(\nu)-1^*$	$\sigma(\nu)-1$	$\sigma(\nu)-1$
3	$\sigma(\nu)$	0	$\sigma(\nu)$	$\sigma(\nu)-1$	$\sigma(\nu)$	0
4	$\sigma(\nu)+1$	1	$\sigma(\nu)+1$	$\sigma(\nu)$	$\sigma(\nu)+1$	1
5	$\sigma(\nu)+2$	$\nu+1$	$\sigma(\nu)+2^*$	$\sigma(\nu)+1$	$\sigma(\nu)+2$	$\sigma(\nu)+2^*$

* indicates that the neighbours are different if the node is on an extremal branch. They belong to another tree: the previous one for B , W_b the next one for W_1 , W_r .

For B and W_b , neighbour 2: $\nu-1$; neighbour 3.

For W_1 and W_r : neighbour 5: $\nu+1$.

In the preferred son tree, the signature of a B -node is always **0**, according of the definition of the tree. The signature of the rightmost son is always **1** and the signature of the leftmost son of a W -node is always **2**.

In the leftmost son tree, the situation is not that clear.

We now turn to the algorithm to compute the branch which leads from the root to the node ν thanks to its coordinate. The algorithms given in the proofs of Theorems 6 and 8 can be simplified in the cases of the tessellations $\{p, 3\}$ and $\{p-2, 4\}$. We apply the same strategy with two auxiliary tables, in order to compute the path from the root to the node whose coordinates constitute the input of the algorithm. The new algorithms are not simply deduced from those of Theorems 6 and 8 as, for instance the digits **1** and **b**₂ which appear there are different while here it is the same digit.

In the case of the preferred son and of the leftmost son trees, the algorithm we produce here for both the pentagrid and for the heptagrid is different from those of Theorems 6 and 8, see Algorithms 5 and 4. In those latter algorithms, the notion of slices was clearly delimited, which is no more the case here. Let $\pi = \mathbf{0}, \dots, \nu_k$ be a path where **0** is the root of the tree. Note that, by definition, in π , ν_{i+1} is a son of ν_i , with $i \in [1..k-1]$ and ν_1 is a son of the root. We say that ν_k is the **end** of π or that π **leads** from the root to ν_k and we write $\pi \models \nu_k$. Say that $k+1$ is the length of π also denoted by $||\pi||$. We say that $\pi = \mathbf{0}, \dots, \nu_i$ with $i \leq k$ is the **beginning** of π up to i and we denote it by $\pi|_i$. We say that

π, μ is a **continuation** of π if μ is a son of ν and only in this case.

LEMMA 13 *Let $\pi \models \nu$ and $\omega \models \nu+1$, with ν and $\nu+1$ on the same level of the tree. We can write $\pi = \mathbf{0}, \dots, \nu_k$ and $\omega = \mathbf{0}, \dots, \mu_k$. Then for $i \in [1..k]$, $\nu_i \leq \mu_i \leq \nu_i+1$. We shall write $\pi \leq \omega \leq \pi+1$ for that relation.*

Proof of Lemma 13. The lemma is true for the sons of the root. Assume it is true for all nodes up to the level n , that level being included. Let ν and $\nu+1$ be both on the level $n+1$. Let μ be the father of ν . If $\nu+1$ is at most the rightmost son of ν , the property is true. The worst case is that ν is the rightmost son of μ . Then, $\nu+1$ is the leftmost son of $\mu+1$. Then, by induction, we have a path $\pi \models \mu$ and a path $\omega \models \mu+1$ satisfying the lemma. Then the paths π, ν and $\omega, \nu+1$ also satisfy the lemma. \square

LEMMA 14 *Consider a node ν in the preferred son tree \mathcal{P} and let be π with $\pi \models \nu$. Then the paths which go to $[\nu]\mathbf{0}$ and to $[\nu]\mathbf{1}$ are continuations of π . Let ω be the path leading to $[\nu]\mathbf{2}$. Let $k = \|\pi\|$. Then, $\|\omega\| = k+1$ and $\pi \leq \omega|_k \leq \pi+1$.*

Proof of Lemma 14. The lemma is true for the root and for its sons and also for the sons of its sons. From the statement of the lemma, we only have to consider the case $[\nu]\mathbf{2}$. Clearly, the signature of ν cannot be $\mathbf{2}$: the pattern $\mathbf{22}$ is ruled out.

Assume that $[\nu] = [\nu_1]\mathbf{0}$. Then, the B -son of ν_1 is $[\nu_1]\mathbf{0}$, its rightmost son $[\nu_1]\mathbf{1}$. The sons of $[\nu_1]\mathbf{0}$ are $[\nu_1]\mathbf{00}$ and $[\nu_1]\mathbf{01}$ so that the sons of $[\nu_1]\mathbf{1}$ are $[\nu_1]\mathbf{02}$, $[\nu_1]\mathbf{10}$ and $[\nu_1]\mathbf{11}$. Hence, $[\nu]\mathbf{2}$ is the leftmost son of $[\nu_1]\mathbf{1}$. Accordingly, If $\pi \models \nu_1$, $\pi, [\nu_1]\mathbf{1}, [\nu_1]\mathbf{02}$ satisfy the assumption of the lemma.

Assume that $[\nu] = [\nu_1]\mathbf{1}$. Whether ν is a B - or a W -node, we know from (21) that its B -son is $[\nu_1]\mathbf{10}$ and its rightmost son is $[\nu_1]\mathbf{11}$ so that $[\nu]\mathbf{2} = [\nu_1]\mathbf{12}$ is the leftmost son of $[\nu_1]\mathbf{2}$ which is $\nu+1$. From Lemma 13, if $\omega \models \nu+1$, writing $\omega = \mathbf{0}, \dots, \nu+1$ and $\pi \models \nu$ with $\pi = \mathbf{0}, \dots, \nu$, we have $\pi \leq \omega \leq \pi+1$, so that $\omega, [\nu_1]\mathbf{12}$ satisfies the conclusion of the lemma. \square

This allows us to justify Algorithm 5.

The idea is to have two paths: $\ell \models [\nu]_i$ and $r \models [\nu]_i+1$ where i is the current position in a loop going down one by one from k to 0, the initialization of ℓ and r being performed in the loop itself. As in Algorithms 3 and 4, the paths are defined by digits in $\{1, 2\}$ from a B -node, in $\{1, 2, 3\}$ from a W -node. An actualization is needed when digits $\mathbf{0}$ or $\mathbf{2}$ are met. We can check it on the lemmas: when $\mathbf{0}$ is met, we make $r|_{i+1} := \ell|_{i+1}$ and $r(i) = \ell(i)+1$, as later we remain in the sub-tree rooted at the node reached by $\ell|_{i+1}$. When $\mathbf{2}$ is met, as the path always goes to right, we have to consider the sons of the node reached by r , so that this time we make $\ell|_{i+1} := r|_{i+1}$ and again $r(i) = \ell(i)+1$.

ALGORITHM 5 *Tessellations $\{7, 3\}$ and $\{5, 4\}$: $p = 7$. Computation of the path from the root to a node in the preferred son tree.*

```

INPUT:  $\mathbf{a}_k.. \mathbf{a}_0$ ;  $\ell, r$ : tables of size  $k+1$ ;  $prev := k$ ;  $s_\ell := W, s_r := W$ ;
  procedure actualize ( $a, b, i, t$ ) is
    begin
      for  $j$  in  $[i..t]$  loop  $a(j) := b(j)$ ; end loop;  $t := i$ ;
    end procedure;
for  $i$  in reverse  $[0..k]$ 
loop if  $\mathbf{a}_i = 0$  --  $i < k$ 
  then actualize ( $r, \ell, i+1, prev$ );
     $\ell(i) := 1$ ;  $r(i) := 2$ ;
    if  $s_\ell = W$  then  $\ell(i) := 2$ ;  $r(i) := 3$ ; end if;
     $s_\ell := B$ ;  $s_r := W$ ;
  elseif  $\mathbf{a}_i = 1$ 
    then if  $i = k$ 
      then  $\ell(k) := 0$ ;  $r(k) := 1$ ;
      else  $r(i) := 1$ ;  $\ell(i) := 2$ ;
        if  $s_\ell = W$ ; then  $\ell(i) := 3$ ; end if;
      end if;
       $s_\ell := W$ ;  $s_r := W$ ;
    else --  $\mathbf{a}_i = 2$ 
      if  $i = k$ 
        then  $\ell(i) := 1$ ;  $r(i) := 2$ ;  $s_\ell := W$ ;  $s_r := B$ ;
        else --  $i < k$ 
          actualize ( $\ell, r, i+1, prev$ );
           $\ell(i) := 1$ ;  $r(i) := 2$ ;  $s_\ell := W$ ;  $s_r := B$ ;
        end if;
      end if;
    end loop;
OUTPUT:  $\ell$ ;

```

Lemma 13 is also true for the leftmost tree: the same argument holds as the tree is build by similar rules. Here, the important fact is that black nodes have one son less than white ones exactly and that each node has one black son exactly. However, Lemma 14 is no more true as stated for the preferred son tree. Here we have:

LEMMA 15 *Consider a node ν in the preferred son tree \mathcal{P} and let be π with $\pi \models \nu$. Then the path which goes to $[\nu]\mathbf{0}$ is a continuation of π . Let ω_α be the path leading to $[\nu]\alpha$ where $\alpha \in \{\mathbf{1}, \mathbf{2}\}$. Let $k = \|\pi\|$. Then, $\|\omega_\alpha\| = k+1$ and $\pi \leq \omega_{\alpha|_k} \leq \pi+1$.*

ALGORITHM 6 *Tessellations $\{7, 3\}$ and $\{5, 4\}$: $p = 7$. Computation of the path from the root to the node in the leftmost son tree.*

```

INPUT:  $\mathbf{a}_k.. \mathbf{a}_0$ ;  $\ell, r$ : tables of size  $k+1$ ;  $prev := k$ ;  $s_\ell := W$ ,  $s_r := W$ ;
  procedure actualize ( $a, b, i, t$ ) is
  begin
    for  $j$  in  $[i..t]$  loop  $a(j) := b(j)$ ; end loop;  $t := i$ ;
  end procedure;
for  $i$  in reverse  $[0..k]$ 
loop if  $\mathbf{a}_i = 0$ 
  then if  $s_\ell = W_0$  or  $s_\ell = W_1$ 
    then actualize( $r, \ell, i+1, prev$ );
       $r(i) := 3$ ;  $s_r := W_1$ ;
    else  $r(i) := 1$ ;  $s_r := B$ ;
    end if;
     $\ell(i) := 2$ ;  $s_\ell := W_0$ ; -- preferred son = 2d son
  elsif  $\mathbf{a}_i = 1$ 
    then if  $s_\ell := W_0$  or  $s_\ell = W_1$ 
      then  $\ell(i) := 3$ ;  $r(i) := 1$ ;  $s_\ell = W_1$ ;  $s_r := B$ ;
      else actualize( $\ell, r, i+1, prev$ );
         $\ell(i) := 1$ ;  $r(i) := 2$ ;  $s_\ell := B$ ;  $s_r := W_0$ ;
      end if;
    else --  $\mathbf{a}_i = 2$ ,  $s_\ell = W_0$  or  $s_\ell = W_1$ ;
      actualize( $\ell, r, i+1, prev$ );
       $\ell(i) := 1$ ;  $r(i) := 2$ ;  $s_\ell := B$ ;  $s_r := W_0$ ;
    end if;
  end loop;
OUTPUT:  $\ell$ ;

```

Proof of Lemma 15. Let $\pi \models \nu$. The fact that the path to $[\nu]\mathbf{0}$ is a continuation of π is a direct corollary of (25).

Consider the case of $[\nu]\mathbf{1}$. If ν is a W_0 or a W_1 -node, the path to $[\nu]\mathbf{1}$ is a continuation of π . If ν is a B -node, $\nu+1$ is in the tree, on the same level of ν . By Lemma 13, there is a path ω leading to $\nu+1$ with $||\omega|| = ||\pi||$ and $\pi \leq \omega \leq \pi+1$. Now, the path to $\nu\mathbf{1}$ is a continuation of ω , so that the lemma is true.

Consider the case of $[\nu]\mathbf{2}$. Then ν cannot be a B -node. Indeed, if ν is a B -node, $\nu+1$ is a W_0 -node. The son signature of $\nu+1$ is $\mathbf{1} \mathbf{0} \mathbf{1}$ while the son signature of ν is $\mathbf{2} \mathbf{0}$, see (25). This means that going from μ , the leftmost son of $\nu+1$, to $\mu+1$ which is the black son of $\nu+1$, we go from a coordinate ending in $\mathbf{1}$ to a coordinate ending in $\mathbf{0}$. This is possible only if $[\mu]$ has a suffix of the form $\mathbf{21}^*$. Now, this pattern is a suffix of $[\nu]$ as $\nu+1$ ends in $\mathbf{0}$. Accordingly, $[\nu]\mathbf{2}$ is not the coordinate of a node of the tree. Hence, ν is a W -node. Whatever the status of $\nu+1$, the signature of its leftmost son is $\mathbf{2}$. Now as $[\nu]$ is not a prefix of the coordinate of the sons of ν , $[\nu]\mathbf{2}$ is the coordinate of the leftmost

son of $\nu+1$. The path to $[\nu]2$ is a continuation of the path to $\nu+1$, so that the lemma is proved here too. \square

This allows us to prove the correctness of Algorithm 6. Again, ℓ and r satisfy $\ell \leq r \leq \ell+1$ at the beginning of the body of the loop and, at the same moment, we have $r(i+1) = \ell(i+1)+1$. These conditions are still true at the end of the body of the loop. Note that the algorithm is a simple translation of the proofs of Lemma 15 and that the actualization is needed when ℓ continues its previous value or when the continuation requires to take r . The actualization is here not symmetric, contrarily to what can be seen in Algorithm 5. Indeed, it was proved in [5] that the path from the root of the tree to a node is the leftmost one in the leftmost son tree. This means that if $\pi \models \nu$ and $\omega \models \nu$ with π in \mathcal{P} and ω in \mathcal{T} , then if $\mu \in \omega$ and $\nu \in \pi$ are on the same level, then $\nu \leq \mu$.

5 Conclusion

These tools offer the possibility to define convenient coordinates for the study of the tessellations $\{p, 3\}$ and $\{p-2, 4\}$. In the case of the pentagrid and of the heptagrid, I used another system based on Fibonacci numbers. The connections between the Fibonacci coordinates and those indicated in this paper, namely in the last part of Section 4. Note that the greatest root of the polynomial (22) is the square of the golden ratio from which the Fibonacci sequence can be obtained.

References

- [1] A. S. Fraenkel, Systems of numerations, *Amer. Math. Monthly*, **92** (1985), 105-114.
- [2] Ch. Frougny, Numeration systems, in *Algebraic Combinatorics on Words*, Cambridge Univ. Press, Cambridge, (to appear), <http://www.igm.uni.-mlv.fr/~berstel/lothaire>.
- [3] M. Hollander, Greedy numeration systems and regularity, *Theory of Comput. Systems*, **31** (1998), 111-133.
- [4] M. Margenstern, *Cellular Automata in Hyperbolic Spaces, vol. 1, Theory*, Collection: *Advances in Unconventional Computing and Cellular Automata*, Editor: Andrew Adamatzky, Old City Publishing, Philadelphia, (2007), 422p.
- [5] M. Margenstern, *Cellular Automata in Hyperbolic Spaces, vol. 1, Theory*, Collection: *Advances in Unconventional Computing and Cellular Automata*, Editor: Andrew Adamatzky, Old City Publishing, Philadelphia, (2007), 422p.

- [6] M. Margenstern, Cellular Automata in Hyperbolic Spaces, Encyclopedia of Complexity and Systems Science, Editor in Chief R.A. Meyers, Springer, (2012), 350-358, doi: 10.1007/978-1-4614-1800-9_24
- [7] M. Margenstern, About embedded quarters and points at infinity in the hyperbolic plane, *arXiv:1507.08495(cs.CG)*, (2015), 17pp.